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Mini-Workshop: Geometries, Shapes and Topologies in PDE-based Applications

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Abstract. The aim of the workshop was to study geometrical objects and their sensitivities in applications based on partial differential equations or differential variational inequalities. Focus topics comprised analytical investigations, numerical developments, issues in applications as well as new and future directions. Particular emphasis was put on: (i) combined shape and topological sensitivity; (ii) extended topological expansions and their numerical realization; (iii) level set based shape and topology optimization.

Mathematics Subject Classification (2000): 49Q10, 49Q20.

Introduction by the Organisers

The Mini-Workshop was jointly organised by Michael Hintermüller, Günter Leugering and Jan Sokolowski.

Shape and topological sensitivities are important tools in many applications in shape design, geometrical optimization and geometrical evolution. Typical applications include the topological optimization of structures, optimization of shapes, geometrical inverse problems e.g. in mathematical image processing, or geometric evolution (like mean curvature flow or other shape gradient related flows). In this context, shape sensitivity typically aims at perturbations of underlying geometrical objects, domains or manifolds, in a direction normal to the given geometry (while keeping the topology of the geometry unchanged), whereas the topological expansion allows to study the sensitivity of a solution of a partial differential equation (PDE) posed on a given domain or manifold and the sensitivity of a geometry dependent objective function with respect to changes of the topology of the underlying geometry, respectively. Such objective functions, occurring for instance in shape optimization, may depend both, directly and implicitly, e.g. through the solution of a (system of) PDEs, on the geometry of interest.

Starting from early work in the field by Murat and Simon, in recent years significant progress has been achieved in the understanding of geometrical objects as variable structures which might be subject to optimization procedures or geometrical evolution. Here we refer to the monographs by Sokolowski and Zolesio as well as Delfour and Zolesio for a summary of the state-of-the-art and further references in shape sensitivity and to work by Sokolowski and Zochovski for basic analytical concepts in topological sensitivity; further see the work by Allaire and Jouve as well as Garreau, Guillaume and Masmoudi where also numerical realizations are presented. Concerning the numerical realization we also mention that level set methods, pioneered by Osher and Sethian, are widely used tools.

Despite the aforementioned progress many important theoretical and numerical questions in shape and topological sensitivity remain widely open. It was therefore the aim of the workshop to bring together a rather diverse group of scientists with all of them being internationally renown researchers in the field and excelling in different branches of the theme of the workshop. Their expertise ranges from analytical investigations, the design and analysis of numerical solution algorithms to the realization and further development of the subject within applications in shape and topology optimization, geometric evolution and geometric inverse problems.

The workshop was attended by 17 participants from 7 countries. In total 13 talks were scheduled in a flexible time frame to allow for ample discussion time, a special lecture (in two parts) by P. Plotnikov on "Compressible Navier-Stokes Equations. Theory and Shape Optimization" related to the recent monograph by P. Plotnikov and J. Sokolowski was scheduled, and a round table discussion on challenges and future topics (including a tutorial on constrained shape optimization) took place on Thursday afternoon.

The various talks of the workshop addressed the following major topics areas:

Combined shape and topological sensitivity. Applications in geometric inverse problems, such as inclusion detection in computerized tomography, for instance, require the combination of topological sensitivities (for an automatized detection of the correct number and topological properties of objects hidden in a given domain from boundary measurements) and shape sensitivity (for adjusting the shape of the detected objects). Currently, the computation of these sensitivities is typically done in separate resulting in two-phase approaches on the numerical level which first apply topological derivatives for detecting inclusions and then shape sensitivities for local shape adjustments. Subsequently, these two phases are repeated until "convergence", i.e., stationarity with respect to both, topology and shape.

Clearly, the separate application of topological and shape derivatives is suboptimal only as one may get stuck at local minimizers possibly far from global optimizers. This behavior is in particular unwanted in the context of geometric inverse problems as the latter typically suffer from numerous (and spurious) local minimizers. Moreover, on the numerical level the two-phase approach requires the separate implementation of the derivatives and their associated induced geometry changes and thus leads to inefficiencies.

A similar need for a combined application of both sensitivity concepts arises in topology optimization (minimal compliance minimization etc.), the design of band gaps in crystals, in Mumford-Shah based image segmentation, as well as in many other applied problems.

Moreover, on the analytical side typically perimeter constraints need to be taken into account in order to have a well-posed shape/topology optimization problem (and to avoid homogenization). From the topological sensitivity point of view this, however, yields singularities due to the difference of dimensionality of the space for the perimeter vs. the change in topology (in the domain). Again, this is a point where both communities need to combine their strengths in order to handle this situation properly on the analytical as well as on the numerical level.

Extended topological expansions and their numerical realization. The current literature almost exclusively considers first order topological expansions only. Indeed, in many applications and in minimal compliance problems in particular, the associated topological gradients yield satisfactory results. Algorithmically, these gradients are realized by "punching" a small hole at the location, where the topological derivative is most negative (if it would be non-negative on the entire domain, then the current shape would be topologically stationary). When creating, at a time, more than one (small) hole in a structure subject to topology optimization, interactions between these holes become important. Such interactions appear to be typically captured by higher order topological expansions only. Further, in the context of geometric inverse problems, there is evidence that extended (beyond first order) topological expansions are indispensable to provide correct information on the location of hidden inclusions. We mention that in this application rather than creating holes in domains, properties of coefficients in PDE-operators are changed in an infinitesimally small ball-shaped domain indicating a change in material properties. The latter obviously also constitute a topological change.

Analytically, such higher order expansions require an improved asymptotic analysis for the solutions of various types of PDEs relevant in associated applications such as the Navier-Lamé system in elasticity, or, in the context of tomography, the Neumann-to-Dirichlet map for second order linear elliptic problems in electrical impedance tomography or Maxwell's system in magnetic induction tomography. Here, higher order expansions of the PDE solution with respect to a characteristic quantity of the considered topological change (e.g., the radius of a hole (elasticity) or an inclusion (tomography)) are needed.

Level set based shape and topology optimization and applications. In many of the aforementioned topics, which were addressed within the workshop, the numerical realization of the shape and topological sensitivity based calculus plays an important role. It is well-known that the level set method, which was popularized by the work of S. Osher and J. Sethian, represents a versatile tool in the numerical realization of moving interface and free boundary problems. Over the years, highly efficient algorithms for the numerical realization of the level set method (narrow band, fast marching method,...) have been invented, analyzed, implemented and used successfully in various applications.

Concerning the combined use of shape gradient related descent methods intertwined with topological sensitivity, level set based techniques are significantly less advanced. In contrast to moving boundary problems, in the context of shape and topology optimization the (pseudo)time marching through the level set equation, a PDE of Hamilton-Jacobi type, when equipped with appropriate descent criteria acts like a line search method well-known from numerical optimization. As a consequence, stability criteria such as the Courant-Friedrichs-Levy (CFL) condition for the discretization in time can be significantly relaxed and, thus, allow for larger geometry changes from one iteration to the next.

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