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Arithmetic Groups vs. Mapping Class Groups: Similarities, Analogies and Differences

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Abstract. Arithmetic groups arise naturally in many fields such as number theory, algebraic geometry, and analysis. Mapping class groups arise in both low dimensional topology and geometric group theory. They have been studied intensively by different groups of people. The purpose of this workshop is to bring experts and aspiring young mathematicians together to interact and develop further exchanges and new collaboration.

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Introduction by the Organisers

Arithmetic groups such as $SL(n, \mathbb{Z})$ occur naturally in many subjects in mathematics such as number theory, representation theory, differential geometry, algebraic geometry and topology. They have been studied intensively. A closely related class of groups consists of mapping class groups of surfaces and has played a fundamental role in the low dimensional topology, algebraic geometry and mathematical physics. Many results about mapping class groups are inspired and motivated by corresponding results on arithmetic groups. For example, it is a known theorem of Borel and Serre that the virtual cohomological dimension of arithmetic groups can be explicitly computed by proving that they are duality groups (i.e., generalized Poincare duality group) in the sense of Bieri and Eckmann. Similar results were then proved by Harer for the mapping class groups.

In proving the above result, the natural action of the arithmetic groups on associated symmetric spaces and the Borel-Serre compactification of the locally symmetric spaces are used crucially. Similarly, the mapping class groups act on the associated Teichmuller spaces, and this action was also used in proving the above result.

The quotient of the Teichmuller space by the mapping class group is the moduli space of Riemann surfaces. In order to construct an analogue of the Borel-Serre compactification of the moduli space, Harvey introduced the notion of curve complex of a surface, which is an analogue of the spherical Tits building of algebraic group and has since played a fundamental role in the recent study of low dimensional topology and mapping class groups.

There are many other analogous results for arithmetic groups and mapping class groups. Furthermore, there are also fruitful interactions between them. For example, there is a Jacobian map from the moduli space of compact Riemann surfaces of genus $g \geq 2$ to the Siegel modular variety of degree g, obtained by associating to its Riemann surface its Jacobian. This Jacobian map has been intensively studied in algebraic geometry. For example, the celebrated Schottky problem is to characterize the image of the Jacobian map. This Jacobian map was used to first prove that the moduli space of Riemann surfaces is a quasi-projective variety. It also allows one to relate topological properties of the above two important classes of groups.

Closely related to the above two classes of groups is the class of outer automorphism groups of free groups. Together they form the most important three classes of groups in geometric group theory.

Some important recent results:

Many important results related to the topics of the proposed workshop have been obtained. We list some of them for a glimpse of the recent status:

(1) The positivity of Gromov norm for irreducible, closed, locally symmetric manifolds with no local $\mathbf{H}^2$ factors, by Lafont and Schmidt in *Simplicial volume of closed locally symmetric spaces of non-compact type*, Acta Math. 197 (2006), no. 1, 129–143.

(2) The proof of the Morita-Mumford-Miller Conjecture on the stable cohomology of the mapping class group by Madsen and Weiss in *The stable moduli space of Riemann surfaces: Mumford's conjecture*, Ann. of Math. (2) 165 (2007), no. 3, 843–941; and its $Out(F_n)$ analogue by Galatius.

(3) The computation of the abstract commensurator of $Out(F_n)$ by Farb and Handel in *Commensurations of* $\mathrm{Out}(\mathrm{F}_n)$, Publ. Math. Inst. Hautes Études Sci. No. 105 (2007), 1–48. This result is analogous to the arithmetic group case by Mostow and Borel and the mapping class group case by Ivanov.

Purpose of the workshop:

The purpose of this workshop is to bring together experts from the following different areas in order to learn from each other and to encourage further interactions between them:

(1) locally symmetric spaces and discrete subgroups of Lie groups, in particular arithmetic groups,
(2) Teichmuller spaces, moduli spaces of Riemann surfaces, and mapping class groups,
(3) Outer automorphism groups of free groups and other closely related groups.
(4) Geometric group theory.

This workshop is the first workshop of such a nature and is well-attended by over 50 people, consisting of both leading experts and aspiring young mathematicians. Most talks are of very high quality and the speakers have tried to make their talks accessible. This is particularly important in view of diversity of participants. All the above topics have been covered. The atmosphere has been very active throughout the workshop, and there have been a lot of interaction and discussion after the talks. Some joint projects between participants have started due to this workshop.

We believe that such a workshop has achieved its goal and will have a lasting impact for various reasons:

(1) Each of the subject has been intensively studied by different groups of people. Many exciting results have been obtained in all these subjects, and it is difficult for any single person to grasp them all.
(2) There have been many analogues and similar results for different classes of groups such as arithmetic groups, mapping class groups and outer automorphism groups. It will be valuable to understand better the underlying unity among them and hence motivate further interactions between them.
(3) In spite of many deep results already obtained, some aspects on interaction between the different groups and spaces described above have not been pursued sufficiently. For example, locally symmetric spaces are special and important examples of complete Riemannian manifolds, and their spectral theory has played a fundamental role in the celebrated Langlands program. On the other hand, the modulis space of curves have not been understood well as Riemannian manifolds, in particular its spectral theory. So far one does not know a natural and complete metric on the outer space yet such that the outer automorphism group $Out(F_n)$ acts isometrically and properly.