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Operator Algebras and Representation Theory: Frames, Wavelets and Fractals

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Abstract. The central focus of the workshop was Kadison-Singer conjecture and its connection to operator algebras, harmonic analysis, representation theory and the theory of fractals. The program was intrinsically interdisciplinary and represented areas with much recent progress. The workshop includes talks on operator theory, wavelets, shearlets, frames, fractals, representations theory and compressed sensing.

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Introduction by the Organisers

Operator algebras, representation theory, and harmonic analysis have always been closely related to physics, in particular quantum theory. Indeed, quantum theory was a driving force behind the early history of operator algebras and representation theory. Early pioneers, such as J. von Neumann, viewed those subjects not as separate fields, but two sides of a general theory. Very recently, other connections to more applied sciences, in particular to engineering, have emerged and stimulated research in mathematics which in turn has led to interdisciplinary work. These connections include wavelet theory, frame theory, fractals, function spaces related to representations, analysis on loop groups, and the geometry of tilings. New connections to approximation theory, numerical mathematics, and microlocal analysis have also influenced today's research atmosphere. Advances made in any one of these subfields will have direct implications for the others. The central

topic of our workshop is an excellent example of recent changes in paradigms and creative interactions. It was built around a single conjecture, the Kadison-Singer conjecture, and yet it involved four separate areas, spanning a wide range from pure to applied mathematics.

Each area was represented by several experts. The meeting therefore was especially fruitful in facilitating discussions across fields, and involving researchers each having unique perspectives, but perhaps not insight all the four main themes.

The Kadison-Singer conjecture was stated in a pioneering 1959-paper by R. V. Kadison, who was a participant at this meeting, and I. Singer [5]. The two authors were inspired in turn by P. A. M. Dirac's use of operators in Hilbert space $\mathcal{H}$: quantum mechanical observables and states. In the theory, pure states are the building blocks. Further Dirac was interested in the role of maximal abelian algebras of bounded operators. The Kadison-Singer conjecture asks if every pure state on the infinite by infinite diagonal matrices (as they are represented by operators on $\mathcal{H}$) extends "uniquely" to a pure state on the algebra of all bounded operators on $\mathcal{H}$. Existence is clear, so the open problem is the uniqueness.

For nearly two decades the Kadison-Singer conjecture (KS) represented only a specialized corner of operator algebra theory. This has all changed. Recently KS has blossomed into a vibrant interdisciplinary research endeavor, with research teams coming from disparate corners of pure and applied mathematics. Our workshop aimed at bringing all of this into focus.

Initially, like with other parts of operator algebras, the KS problem had its roots in quantum mechanics. More recently, two things happened to bring KS back front and center, as a common thread in four fields, some of them often thought of as disparate: (i) operator algebras, representation theory and harmonic analysis, (ii) signal processing, and the use of frames (over-complete "bases"), (iii) combinatorics of pavings, and (iv) Banach space geometry, see e.g., [3]. The four themes in turn overlap with other research trends, one being multi-scale theory. This lies behind powerful tools used both in frames and in the analysis of fractals. An important class of frames constitutes wavelet families built from scale similarity (hence multi-scale). By their very definition, fractals are understood from the similarity of data or geometries at different scales.

One breakthrough was papers [1, 2] by Joel Anderson which established the equivalence between KS and what is now called the paving conjecture; and the other was work [4] initiated by P. Casazza, who also attended the workshop, showing that KS is equivalent to key conjectures in signal and image processing, one of them known as the Feightinger conjecture. These conjectures are, on the face of it, from quite disparate areas of mathematics. It is therefore especially intriguing that, quite recently, their equivalence has been established. This also means that the resolution of the conjecture is now more likely, and that the answer will have implications going far beyond just finding out if the answer is "yes" or "no", The implications of a no-answer will be as important as those deriving from an affirmative resolution.

This diversity of topics closely connected to the Kadison-Singer conjecture is illustrated by the listed topics below, and by speakers at the meeting. Each talk was interdisciplinary in the sense of giving rise to a lively discussion between participants from diverse areas. The topics covered included:

(1) Operator Algebras, including various aspects of the Kadison-Singer conjecture and, related to this, extensions of the Schur-Horn theorem and the Bourgain-Tzafriri restricted invertibility theorem.
(2) Representation Theory, relating Feichtinger's conjecture to duality principles.
(3) Frame Theory, with topics ranging from spanning and linear independence properties over semiframes to duality principles.
(4) Applied Harmonic Analysis, focusing in particular on the novel anisotropic system of shearlets and on various aspects of uncertainty principles.
(5) Fractal Theory, in particular, analysis on Cantor sets.
(6) Classical Harmonic Analysis and Sampling Theory, covering diverse aspects of sampling from the Zak transform to extensions of Shannon's sampling theorem.
(7) Compressed Sensing and Sparse Approximation, including the theory related to the Johnson-Lindenstrauss lemma.

During the meeting 21 expert talks were presented. To also accommodate younger researchers, we had one session of shorter talks.

In addition to the Oberwolfach Reports with Abstracts of talks (edited by J. Lemvig in collaboration with the organizers), we have arranged a journal special issue in the journal *Numerical Functional Analysis and Optimization* to help disseminate results from our workshop. This will include refereed papers also from participants who did not get a chance to deliver formal talks at the workshop. The editors are Pete Casazza, Palle Jorgensen, Keri Kornelson, Gitta Kutyniok, David Larson, Peter Massopust, Gestur Ólafsson, Judith Packer, Sergei Silvestrov, and Qiyu Sun.

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