MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

Representation Theory of Quivers and Finite Dimensional Algebras

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ABSTRACT. Methods and results from the representation theory of quivers and finite dimensional algebras have led to many interactions with other areas of mathematics. Such areas include the theory of Lie algebras and quantum groups, commutative algebra, algebraic geometry and topology, and in particular the new theory of cluster algebras. The aim of this workshop was to further develop such interactions and to stimulate progress in the representation theory of algebras.

Introduction by the Organisers

The representation theory of quivers is probably one of the most fruitful parts of modern representation theory because of its various links to other mathematical subjects. This has been the reason for devoting a substantial part of this Oberwolfach meeting to problems that can be formulated and solved involving quivers and their representations. The interaction with neighbouring mathematical subjects like geometry, topology, and combinatorics is one of the traditions of such Oberwolfach meetings; it can be quite challenging for the participants but it certainly continues to be a source of inspiration. There were 27 lectures given at the meeting, and what follows is a quick survey of their main themes.

Representations of quivers. There continues to be rapid development in the theory of representations of quivers, especially in the following interlinked areas:

the geometry of quiver varieties (in the sense of Nakajima, so moduli spaces of representations of the double of a quiver, with relations coming from a moment map, as in the preprojective algebra), moduli spaces of representations of an undoubled quiver, Hall algebras, and Donaldson-Thomas invariants for quivers.

Three of the speakers, T. Hausel, E. Letellier and F. Rodriguez Villegas, spoke on developments arising from their work on arithmetic harmonic analysis on character and quiver varieties. T. Hausel spoke about a series of conjectures relating the cohomology of character varieties with Kac's A-polynomial, which counts the number of absolutely indecomposable representations of a quiver over finite fields of varying sizes. One aim of the conjectures is to find a cohomological proof of Kac's conjecture that the coefficients of the A-polynomial are non-negative. Mostly, the quivers which arise this way are of a specific shape - 'comet-shaped' - but one conjecture extends these ideas to arbitrary quivers. E. Letellier's talk related to the character theory of the general linear group over a finite field. Although the irreducible characters are known since Green's work in 1955, the decomposition of tensor products is not understood. By linking this problem with intersection cohomology of quiver varieties, E. Letellier was able to show in some cases, when the characters are sufficiently generic, that whether or not a given irreducible occurs in a tensor product is determined by a Kac-Moody root system. Rodriguez Villegas explored a refinement of Kac's A-polynomial, and presented an explicit formula for its evaluation at 1 in case the quiver consists of one vertex and several loops.

M. Reineke and S. Mozgovoy both spoke about Donaldson-Thomas invariants associated to symmetric quivers. Reineke described an explicit combinatorial treatment of DT invariants for the quiver with one vertex and several loops. A conjecture of Kontsevich and Soibelman (now proved by Efimov, which we learnt about during the workshop, and with a preprint subsequently posted to the arxiv) implies a positivity property for DT invariants, and S. Mozgovoy showed that this property implies Kac's non-negativity conjecture (mentioned above) for quivers with a loop at every vertex. It is amazing that such different approaches can lead to progress with this conjecture!

Instead of passing to a moduli space of representations of a quiver, one can consider the corresponding group action on the affine space of representations of a fixed dimension vector. Associated to any irreducible closed subset, stable under the group action, there are classes in both equivariant cohomology and K-theory. Such classes have been studied by Buch and others as a way to generalize and unify various notions in Schubert calculus. The classes can be written as linear combinations of products of Schur and Grothendieck polynomials, and in his talk A. Buch described conjectural properties of the coefficients in these linear combinations (known as 'quiver coefficients'), and results obtained in case the quiver is of Dynkin type.

There are many very basic open questions about representations of quivers, and one such is how to construct the indecomposable representations in general. However in his talk, T. Weist showed that for every dimension vector which is an

imaginary Schur root, there exists an indecomposable representation given by a tree.

Cluster algebras and cluster categories. Cluster algebras are certain commutative algebras whose generators and relations are constructed recursively. They were invented by S. Fomin and A. Zelevinsky in the year 2000 to serve as a combinatorial framework for the study of Kashiwara/Lusztig's canonical bases in quantum groups and of the closely related notion of total positivity in algebraic groups. More than a decade after Fomin-Zelevinsky's invention, the precise connection between cluster algebras and canonical bases remains a mystery. The best results confirming that such a link exists are certainly those due to Geiss-Leclerc-Schröer, who have shown that in the cluster algebras arising as rings of coordinates on unipotent cells in Kac–Moody groups, all cluster monomials belong to Lusztig's dual semi-canonical basis. In his talk, C. Geiss presented an interpretation of the dual semi-canonical basis as the 'generic basis', which allows its conjectural generalization to arbitrary cluster algebras. These remarkable results were complemented in P.-G. Plamondon's talk, devoted to a mutation-invariant parametrization of the elements of the conjectural generic basis in an arbitrary cluster algebra. This parametrization yields an important connection between Geiss-Leclerc-Schröer's conjecture and Fock-Goncharov series of duality conjectures motivated by their higher Teichmüller theory. It is expected that a quantum version of the generic basis will yield a generalization of Kashiwara/Lusztig's canonical basis to an arbitrary cluster algebra. The very first results in the direction of this long-term goal are due to P. Lampe, who, in his talk, explained how in type A, the quantum cluster algebra identifies with a quantum coordinate algebra in such a way that the quantum cluster variables correspond to certain canonical basis vectors.

In their 2008 preprint 'Stability structures, Donaldson-Thomas invariants and cluster transformations', Kontsevich-Soibelman have interpreted individual cluster transformations as wall-crossing formulas for DT-invariants of certain 3-Calabi-Yau categories. In remarkable work, K. Nagao has extended their idea to compositions of cluster transformations and combined it with D. Joyce's results to prove a series of conjectures formulated by Fomin-Zelevinsky in 2006 (and recently proved using different methods first by Derksen-Weyman-Zelevinsky and then by Plamondon). Nagao's talk combined a streamlined introduction to Donaldson-Thomas theory with a beautiful presentation of the key ideas of his approach.

Cluster categories are triangulated 2-Calabi-Yau categories used to 'categorify' cluster algebras (as in the talks by Geiss and Plamondon). The study of particular classes of such categories reveals intricate combinatorial structures. For finite cluster type and for tubes, these were explored in the talks by T. Holm and by K. Baur. For cluster categories associated with 'ciliated surfaces', R. Marsh analyzed the combinatorics of the mutation of rigid objects in terms of coloured quivers. Generalized cluster categories of higher Calabi-Yau dimension were at the center of O. Iyama's talk. After a beautiful introduction to this circle of ideas he sketched an extension of his 'higher Auslander-Reiten theory' to the new

class of 'n-representation-controlled algebras' (joint work with S. Oppermann and
M. Herschend).

Derived categories and tilting theory. Much of the recent progress in repre-
sentation theory of algebras is formulated in terms of derived categories. In fact,
the derived category of an algebra captures a wealth of homological information
and is an interesting invariant in its own right. The study of the existence and
properties of (cluster) tilting objects provides one of the challenges in this sub-
ject. The talks of A. Beligiannis and L. Hille were devoted to this aspect. M. Van
den Bergh talked about autoequivalences of derived categories for singular elliptic
curves and pointed out the connection with mirror symmetry. Another method
to approach a derived category is the use of stratifications. In Koenig's talk, the
stratification of the derived category of an algebra was discussed in terms of rec-
ollements. A completely different way of stratifying the stable module category
of a finite group was explained in S. Iyengar's talk. He used group cohomology
and presented the connection with specific properties of the Bousfield lattice. An
interesting numerical invariant of a derived category is its dimension as a trian-
gulated category. It is a somewhat surprising result presented by S. Oppermann
that this dimension is finite for the derived category of finite dimensional modules,
while any proper triangulated subcategory containing the projectives is of infinite
dimension.

Auslander-Reiten theory. One classical invariant in the representation theory
of artin algebras is the Auslander-Reiten quiver of the category of the finitely
generated (left or right) modules. The quiver records the isomorphism classes of
the indecomposable modules and their relative position with respect to the radi-
cal of the category. In fact it records the category of finitely generated modules
modulo the infinite radical, $\mathrm{rad}^\omega = \cap_{i \geqslant 0} \mathrm{rad}^i$. The talk of C. M. Ringel dis-
cussed how to describe the module category of finitely generated modules modulo
$\mathrm{rad}^{\omega 2}$ for suitable (1-domestic) special biserial algebras. This is obtained through
the socalled Auslander-Reiten quilt of the algebra, which is the Auslander-Reiten
quiver with additional vertices inserted for indecomposable algebraically compact
infinite dimensional modules and a convergence relation. The talk illustrated this
construction through a careful study of one example.

   The Auslander-Reiten quiver gives rise to different classes of modules and in-
variants of the algebra. One important notion is that of a module lying on a short
chain: Given an almost split sequence $0 \to A \to B \to C \to 0$, the end terms are
connected via the Auslander-Reiten translate τ, that is, $A \simeq \tau(C)$. Then an inde-
composable module M is *on a short chain* if there are non-zero homomorphisms
$X \to M \to \tau(X)$ for some indecomposable module X. An interesting fact about
indecomposable modules not in the middle of a short chain is that they are deter-
mined up to isomorphism by their composition factors. An omnipresent class of
algebras in the representation theory of artin algebras is the *tilted* algebras, that
is, endomorphism rings of a tilting module over a hereditary algebra. In the talk

of A. Skowronski tilted algebras were characterized by the existence of a sincere module which is not in the middle of a short chain.

Some links to commutative algebra. Eisenbud pointed out early on a link between complete intersections and group rings by showing that ideas from homological algebra for complete intersections transferred to group rings by studying projective resolutions and in addition introducing matrix factorizations. This approach was extended in the talk of R.O. Buchweitz to construct complete resolutions and showing that maximal Cohen-Macaulay modules over complete intersections are also determined by matrix factorizations, not over the original ring, but over a naturally associated larger ring.

In the mid 1980's there was a strong influence from representation theory on commutative algebra, through Auslander-Reiten theory for maximal Cohen-Macaulay modules over isolated singularities and a theory of homologically finite subcategories. The talk of J. Weyman extended this interplay as it dealt with a new connection between generic free resolutions in commutative algebra and Kac-Moody Lie algebras. A free complex

$$0 \to F_n \to F_{n-1} \to \cdots F_1 \to F_0 \to 0,$$

has format $(r_n, r_{n-1}, \ldots, r_1)$ over a commutative ring if the rank of the i-th differential d_i equals to r_i for all i. Then an acyclic complex F_{gen} over a given ring R_{gen} is *generic* if for every complex G of a given format $(r_n, r_{n-1}, \ldots, r_1)$ over a Noetherian ring S there exists a homomorphism $f \colon R_{gen} \to S$ such that $G \simeq F_{\mathrm{gen}} \otimes_{R_{\mathrm{gen}}} S$. For $n = 3$ associate to the format (r_3, r_2, r_1) a graph $T_{p,q,r}$ with three arms of length $p = r_3$, $q = r_2 - 2$ and $r = r_1$. Then Weyman showed, among other things, that there exists a Noetherian generic ring for this format if and only if the graph $T_{p,q,r}$ is Dynkin. For the case $n = 2$ the problem was solved by Hochster and Huneke.

Further links to Lie theory and algebraic groups. M. Brion spoke on a representation-theoretic approach to the study of homogeneous bundles on abelian varieties. This allowed him not only to recover a classical structure theorem of Miyanishi and Mukai (in characteristic zero) but also to obtain new results on projective bundles, which he linked to standard representations of Heisenberg groups.

C. Stroppel presented ongoing joint work with E. Frenkel and J. Sussan motivated by the problem of categorifying Turaev-Viro invariants of 3-manifolds. A key step is the categorification of tensor products of representations of quantum groups and of the intertwiners between tensor products. She focused on the case of sl_2, where she showed that a categorification with excellent properties is provided by a certain category of Harish-Chandra bimodules. By work of Futorny-Mazorchuk-König, this category is properly stratified, which provides a beautiful link to the representation theory of finite-dimensional algebras.

The format of the workshop has been a combination of introductory survey lectures and more specialized talks on recent progress. In addition there was plenty of time for informal discussions. Thus the workshop provided an ideal atmosphere

for fruitful interaction and exchange of ideas. It is a pleasure to thank the administration and the staff of the Oberwolfach Institute for their efficient support and hospitality.