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Mini-Workshop: Mathematical Analysis for Peridynamics

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Abstract. A mathematical analysis for peridynamics, a nonlocal elastic theory, is the subject of the mini-workshop. Peridynamics is a novel multiscale mechanical model where the canonical divergence of the stress tensor is replaced by an integral operator that sums forces at a finite distance. As such, the underlying regularity assumptions are more general, for instance, allowing discontinuous and non-differentiable displacement fields. Although the theoretical mechanical formulation of peridynamics is well understood, the mathematical and numerical analyses are in their early stages. The mini-workshop proved to be a catalyst for the emerging mathematical analyses among an international group of mathematicians.

Mathematics Subject Classification (2000): 35S11, 70G70, 70G75, 74B20.

Introduction by the Organisers

The mini-workshop *Mathematical Analysis for Peridynamics*, organised by Etienne Emmrich (Bielefeld), Max Gunzburger (Tallahassee), and Richard Lehoucq (Albuquerque), was held January 16th–January 22nd, 2011. This meeting was attended by 17 participants with broad geographic representation.

The response of materials to environments and loads occurring in practice requires an understanding of mechanics at disparate spatial and temporal scales. Such "multiscale" understanding is a fundamental challenge for next generation materials modeling. A currently popular multiscale approach couples two or more well-known models, for example, molecular dynamics and classical elasticity, each of which is useful at different scales. Although some notable successes have resulted from this type of multiscale material modeling, some issues remain unresolved, some of which stem from the inherent difficulty encountered when coupling local models to nonlocal ones.

An alternative approach is to develop a single multiscale material model that remains valid and useful over a wide range of temporal and spatial scales. Peridynamics [5], a nonlocal elastic theory, is a promising multiscale mechanical nonlinear model. The canonical divergence of the stress tensor is replaced by an integral operator that sums forces at a finite distance. As such, the underlying regularity assumptions are more general, for instance, allowing discontinuous, let alone non-differentiable, displacement fields. For example, the recent review [1] includes peridynamic applications to fracture and failure of composites, nanofiber networks, and polycrystal fracture. The article [7] studies the peridynamic model for solid mechanics. Furthermore, although peridynamics by itself is a multiscale material model and has proved to be extremely useful for simulations of singular phenomena such as fracture, peridynamics also can be used as bridge between local continuum models and nonlocal atomistic models, mitigating the difficulties one encounters when trying to directly couple the latter two types of models.

The goal of the mini-workshop is to bring together applied and computational mathematicians, and mechanicians to further the mathematical understanding of peridynamics. Although the theoretical mechanical formulation of peridynamics is well understood, the mathematical and numerical analyses are in their early stages (see [2, 3, 4, 6] for examples). Successful mathematical treatments of peridynamics are not only interesting from the mathematics point of view, but will lead to improved temporal and spatial multiscale discretization and solution algorithms, and improved understanding of the range of applicability of peridynamics. Topics of interest include:

- well posedness of the time-dependent peridynamics equation of motion; Nonlocal vector calculus, variational formulations of peridynamic models; homogenization; stochastic peridynamic models;
- analysis and development of a finite element and other discretization methods; development and analysis of efficient and robust solution methods for discretized peridynamic models; coupling peridynamics to molecular dynamics and finite element discretizations of classical elasticity;
- relationship and convergence to classical elasticity as the nonlocality vanishes; relationship with other nonlocal continuum mechanical theories.

References

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