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Mini-Workshop: Dynamics of Trace Maps and Applications to Spectral Theory

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Abstract. Recently some exciting results for the spectrum of Schrödinger operators with self-similar potentials and the dynamics of the associated Schrödinger equation have been established using input from complex and smooth dynamics. The workshop allowed us to bring together experts in dynamical systems, spectral theory, and quasi-crystals to study these deep and promising relations.

Mathematics Subject Classification (2000): 37D20, 47B36, 81Q10.

Introduction by the Organisers

In the early 1980's, two groups independently laid the groundwork for much activity in (solid state) physics and mathematics (spectral theory) in the decades since then. Shechtman et al. discovered new structures, nowadays called quasi-crystals, that have unexpected and intriguing behavior. Namely, these structures have a diffraction pattern resembling that of crystals but also displaying rotational symmetries that are impossible for crystals. Kohmoto et al. on the other hand proposed a simple quasi-periodic Schrödinger operator, the Fibonacci Hamiltonian, with critical behavior for all non-zero values of the coupling parameter. That is, the eigenfunctions are neither localized nor extended, the spectral measures are purely singular continuous, and quantum transport is anomalous. All these properties are by now rigorously established.

The mathematical models of quasi-crystals have a number of features, such as being constructed by a cut-and-project method (projection of a part of a higher-dimensional lattice along incommensurate directions), displaying self-similarity (resulting from a construction based on inflation), and of course the desired diffraction behavior. The model proposed by Kohmoto et al. has all these features and it has consequently become the standard one-dimensional quasi-crystal model.

The mathematics involved in the analysis of such models has many facets. Naturally, it involves discrete geometry, spectral theory, and harmonic analysis. In addition, dynamical systems play an important role. As shown by Kohmoto et al., the self-similarity gives rise in a natural way to a renormalization procedure which results in a direct correspondence between the desired spectral properties and the dynamics of a three-dimensional polynomial map, $T(x, y, z) = (2xy-z, x, y)$. Since the variables correspond to traces of transfer matrices, the map T is called the *trace map*. In fact, any self-similar model gives rise to a corresponding polynomial map and hence there is a rich class of trace maps whose dynamics are directly related to a certain class of Schrödinger operators. Moreover, since the spectra of these Schrödinger operators are zero-measure Cantor sets, methods from geometric measure theory have also been applied successfully to study the scaling properties of these Cantor sets.

Trace maps have been studied with a view towards spectral theory in many works. Most of them regard T as a real dynamical system, that is, as $T : \mathbb{R}^3 \to \mathbb{R}^3$. This is in some sense natural since due to self-adjointness of the Schrödinger operators, spectra and spectral measure live on $\mathbb{R}$ and for real energies, all traces are real. However, some recent works have challenged this picture and instead studied $T : \mathbb{C}^3 \to \mathbb{C}^3$. This innocent-looking change of perspective has opened up a new tool-box, that of complex analysis and complex dynamics, and in fact allowed one to prove spectral results that had been completely out of reach.

Another promising new fusion of ideas involves the application of uniformly and normally hyperbolic dynamics to trace maps and has led to a multitude of new results for the weakly coupled Fibonacci Hamiltonian. Among these results are estimates of the fractal dimension of the spectrum and complete gap labeling in the sense of Bellissard.

The aim of the workshop was to pursue these new points of view vigorously.

The structure of the workshop was the following. The meeting was attended by 17 participants, with each presenting their results and/or a historical overview of the subject. We started off with several overview lectures (by Kohmoto, Sütő, Bellissard, Grimm, Damanik), and continued with presentations of particular results. One of the talks (by Lifshitz) provided an exposition of physical point of view, which was quite inspiring for mathematicians. In the very last talk Embree presented numerical results on spectral properties of Fibonacci Hamiltonian, which also suggest numerous new conjectures. One night Grimm presented a beautiful general-audience talk, entitled “A hexagonal monotile for the Euclidean plane,” which many participants from the other mini-workshops also attended.

We have the feeling that bringing together people from dynamical systems, spectral theory, mathematical physics, and physics of quasicrystals turned out to be amazingly productive, provided deeper understanding of the subject by all the participants, and will eventually lead to new exciting results.