Abstract. The workshop on Low-Dimensional Topology and Number Theory brought together researchers in these areas with the intent of exploring the many tantalizing connections between Low-Dimensional Topology and Number Theory. Some of the most actively discussed topics were the appearances of modularity in quantum invariants and mutual relations between hyperbolic volume, K-theory, and asymptotics of quantum invariants.

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Introduction by the Organisers

The workshop Low-Dimensional Topology and Number Theory, organized by Paul E. Gunnells (Amherst), Walter Neumann (New York), Adam S. Sikora, (Buffalo), Don Zagier (Bonn/Paris) was held August 15th – August 21st, 2010. This meeting brought together 45 researchers from around the world, at different stages of their careers – from graduate students to some of the most established scientific leaders in their areas. The participants represented diverse backgrounds from low-dimensional topology, number theory, and quantum physics.

The workshop consisted of 28 talks ranging from 30 to 50 minutes intertwined with informal discussions. As suggested by its name, the workshop was devoted to the connections between Low-Dimensional Topology and Number Theory. Some of the most actively discussed topics are elaborated below.

0.1. Arithmetic Topology. Starting from the late 1960’s, B. Mazur and others observed the existence of a curious analogy between knots and prime numbers
and, more generally, between knots in 3-dimensional manifolds and prime ideals in algebraic number fields. For example, the spectrum of the ring of algebraic integers in any number field has étale cohomological dimension 3 (modulo higher 2-torsion). Moreover, the étale cohomology groups of such spectra satisfy Artin-Verdier duality, which is reminiscent of the Poincare duality satisfied by closed, oriented, 3-dimensional manifolds. Furthermore, there are numerous analogies between algebraic class field theory and the theory of abelian covers of 3-manifolds. As an example we mention that the universal abelian cover surprisingly yields complete intersections on both the number-theoretic (De Smit–Lenstra) and topological/geometric (Neumann–Wahl) sides. However, there is no good understanding of the underlying fundamental reasons for the above analogies, and of the full extent to which these analogies hold.

0.2. Arithmetic of Hyperbolic Manifolds. A hyperbolic 3-manifold $M$ is an orientable 3-dimensional manifold that is locally modeled on hyperbolic three-space $\mathbb{H}^3$. Its fundamental group is a torsion-free discrete subgroup $\Gamma \subset PSL_2(\mathbb{C})$ defined uniquely (up to conjugation) by the requirement $M = \Gamma \backslash \mathbb{H}^3$. According to the geometrization program of Thurston, which has been the guiding force in 3-manifold topology research since the 1980’s, hyperbolic 3-manifolds constitute the largest and least understood class of 3-manifolds. Such manifolds have been studied from a variety of perspectives, including geometric group theory, knot theory, analysis, and mathematical physics.

Recently, number theory has played a particularly crucial role in the study of hyperbolic 3-manifolds. The connection arises when one assumes that $M$ has finite volume. In this case, Mostow rigidity asserts that the matrices representing $\Gamma \subset PSL_2(\mathbb{C})$ can be taken to have entries in a number field. A consequence of this connection is that techniques from algebraic number theory now play a central role in the experimental investigation of hyperbolic 3-manifolds, to the extent that the premier software for performing computations with hyperbolic 3-manifolds (SNAP) incorporates the premier software for experimental algebraic number theory (GP-PARI)! Furthermore, via this connection low-dimensional topology and number theory inform each other in surprising ways. For example, volumes of hyperbolic manifolds can be expressed in terms of special values of Dedekind zeta functions (Borel).

0.3. Connections with algebraic $K$-theory. To a hyperbolic 3-manifolds one can associate a canonical element in algebraic $K$-theory, more specifically in $K_3(\bar{\mathbb{Q}})$. Thanks to work of Neumann–Yang and more recent work of Zickert, this element can be completely computed from any triangulation of the 3-manifold. Many invariants are related to, or even determined by, this class, the most prominent being the hyperbolic volume, which is given by the Bloch–Wigner dilogarithm function. This leads to a variety of connections with number theory. Some of these have to do with the asymptotics and modularity discussed below. Another connection is the complex of questions connected with the Mahler measure. This is an
invariant of polynomials that appears in surprisingly diverse areas of mathematics, for example, as the entropy of certain dynamical systems and as the value of higher regulators (Beilinson, Boyd, Rodriguez Villegas), and the theory of heights and Lehmer’s conjecture. But more surprisingly, they appear in the theory of hyperbolic 3-manifolds: the Mahler measures of polynomials appearing naturally in 3-dimensional topology (such as $A$-polynomials, Alexander polynomials, and Jones polynomials) give interesting examples from the perspective of arithmetic.

0.4. **Quantum topology, asymptotics, modular forms.** Further connections between quantum topology and number theory appear in the context of quantum invariants of 3-manifolds. These invariants are typically functions that are defined only at roots of unity, and one can ask whether they extend to holomorphic functions on the complex disk with interesting arithmetical properties. A first result in this direction was found by R. Lawrence and D. Zagier, who showed that for certain 3-manifolds (e.g. the Poincaré homology sphere) the quantum invariants at roots of unity are limiting values of Eichler integrals of certain modular forms. More recently Zagier has found experimental evidence of modularity properties of a new type for the Kashaev invariants of knots. The outstanding open problem concerning Kashaev invariants, the so-called Volume Conjecture (which relates the value at $e^{2\pi i/N}$ of the $N$th Kashaev invariant of a knot to its hyperbolic volume), also has a beautiful arithmetic refinement in which the Kashaev invariants have asymptotic expansions around all roots of unity that have algebraic coefficients (in the trace field of the knot) and are related by modular transformations. Testing these conjectures is a challenging computational problem that now finally seems in reach.

0.5. **Multiple zeta values.** Although originally defined analytically, Kontsevich’s universal finite type invariant of links can be calculated algebraically, by an application of the Drinfeld associator, a certain formal infinite power series on two non-commuting variables. It turns out that the coefficients of this series are given by special values of the Riemann zeta function and the multiple zeta functions of Euler and Zagier. By choosing different diagrams representing the same link $L$, one obtains different expressions for the Kontsevich invariant of $L$. Equating these expressions then yields non-trivial identities for the special values of these zeta functions. This topic was not lectured on at the meeting; nevertheless, it remains of considerable interest from the point of view of both number theory and topology.