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Arbeitsgemeinschaft: Ricci Flow and the Poincare Conjecture

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Abstract. It was the aim of this workshop to introduce the participants to the basic concepts and techniques of Hamilton's Ricciflow programme and cover the main ideas of the proof of the Poincaré conjecture. This was accomplished by having participants present segments of the proof, where an effort was made to include as much detail as possible.

Mathematics Subject Classification (2000): 53C44, 57M40.

Introduction by the Organisers

The Ricciflow, introduced by Richard Hamilton [Ha1], is a geometric evolution equation which deforms the metric on a Riemannian manifold smoothly in the direction of its Ricci curvature. More precisely, the evolution equation for the family of metrics (g_{ij}) on a manifold M is given by

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}, \tag{0.1}$$

where R_{ij} denotes the Ricci tensor corresponding to the metric. Written in suitable local coordinates this equation has the form of a nonlinear heat type equation for the metric symbols. Because of this one might naively expect that the equation will try to evolve the geometry on M to one which looks the same at every point on the manifold, a homogeneous geometry. This intuition is correct in dimension two, where the Ricciflow can be used [Ha2], [Ch] to conformally deform any metric on a closed surface to one of constant curvature, which provides a new proof of the famous uniformization theorem.

In higher dimensions, however, the geometry will in general become singular in finite time, i.e. the norm of some of the sectional curvatures will tend to infinity at certain points on the manifold. In three dimensions, if the initial metric has positive Ricci curvature and M is closed and simply connected, Hamilton [Ha1] showed that, after suitable rescaling of the evolving metric, such as to keep the volume of the manifold constant, the metric tends smoothly to the metric on the standard S^3.

Soon after that, Hamilton [Ha3] set up a programme which had the aim of settling Thurston's geometrization conjecture using Ricciflow. This conjecture asks whether any closed 3-manifold can be decomposed along 2-spheres and incompressible tori in such a way that after capping of the 2-sphere boundaries by 3-balls, the resulting finitely many geodesically complete pieces would each admit one out of a list of eight homogeneous geometries formulated by Thurston [Th]. In particular, this would prove the famous Poincaré conjecture, that any simply connected orientable closed 3-manifold had to be topologically equivalent to S^3.

Hamilton himself, but also many others, completed many of the crucial steps in this programme (see [CLN]) but several severe technical difficulties remained unsettled for at least one decade. In 2002, Perelman [P1] - [P3] introduced a number of completely novel ideas and techniques that eventually led to the resolution of the geometrization and hence also the Poincaré conjecture.

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