MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

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Classical Algebraic Geometry

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ABSTRACT. Algebraic geometry studies properties of specific algebraic varieties, on the one hand, and moduli spaces of all varieties of fixed topological type on the other hand. Of special importance is the moduli space of curves, whose properties are subject of ongoing research. The rationality versus general type question of these spaces is of classical and also very modern interest with recent progress presented in the conference. Certain different birational models of the moduli space of curves have an interpretation as moduli spaces of singular curves. The moduli spaces in a more general setting are algebraic stacks. In the conference we learned about a surprisingly simple characterization under which circumstances a stack can be regarded as a scheme. For specific varieties a wide range of questions was addressed, such as normal generation and regularity of ideal sheaves, generalized inequalities of Castelnuovo-de Franchis type, tropical mirror symmetry constructions for Calabi-Yau manifolds, Riemann-Roch theorems for Gromov-Witten theory in the virtual setting, cone of effective cycles and the Hodge conjecture, Frobenius splitting, ampleness criteria on holomorphic symplectic manifolds, and more.

Introduction by the Organisers

The Workshop on Classical Algebraic Geometry, organized by David Eisenbud (Berkeley), Joe Harris (Harvard), Frank-Olaf Schreyer (Saarbrücken) and Ravi Vakil (Stanford), was held June 8th to June 14th. It was attended by about 45 participants from USA, Canada, Japan, Norway, Sweden, UK, Italy, France and Germany, among of them a large number of strong young mathematicians. There were 18 talks with a maximum of four talks per day, which left plenty of time for lots of informal discussions among smaller groups.

A widespread variety of themes in classical algebraic geometry were discussed, many of them with a very modern point of view. Although it is tempting to summarize each of the talks, we limit ourselves to four highlights:

- Mihnea Popa reported on ongoing work with Lazarsfeld and Pareschi, in which they use the Bernstein-Gel'fand-Gel'fand correspondence to obtain very geometric results. The BGG correspondence is between an exterior module and a linear complex of vector bundles on projective space. Popa considers the cohomology algebra of the sheaf of functions, which is naturally an exterior module over the exterior algebra generated by the first cohomology group. Using recent results of Eisenbud, Floystad, and Schreyer, they translate results from the linear complex of vector bundles back to the geometric setting. For example, if X is a compact Kähler manifold with generically finite Albanese map, then the canonical module is generated in degree 0, and has a linear free resolution over the exterior algebra. In the case when X has no higher irrational pencils, the complex on $\mathbb{P}^n$ is actually a linear free resolution of a vector bundle. Its short length imposes, via vector bundle techniques, bounds on the Betti numbers of such manifolds, extending and improving well-known results in the case of surfaces.
- In ideal circumstances, objects in algebraic geometry are parametrized by geometric spaces as we are used to working with them: varieties and schemes. However, even in very simple circumstances, such as curves, this is too much to hope for. Mumford's Geometric Invariant Theory is a good approximation in a precise sense. Then, in 1969, Deligne and Mumford developed what are now known as Deligne-Mumford stacks, a new class of geometric objects that include many important moduli spaces. They have an underlying geometric space, known as the coarse moduli space. However, objects with infinitely many automorphisms (such as vector bundles) don't admit such moduli spaces, and the more general notion of Artin stack is often needed. This is seen by much of the community as a difficult, non-geometric concept. But in Jarod Alper's talk, he introduced the notion of a *good moduli space* for Artin stacks, which is an honest scheme. The definition is remarkably short: a map $\mathcal{X} \to X$ from an Artin stack is a *good moduli space* if it is cohomologically affine (pushforward of quasicoherent sheaves is exact) and if the pushforward of the sheaf of functions on $\mathcal{X}$ is the sheaf of functions on Y. From this simple definition, a number of wonderful properties readily follow, and indeed the theory of GIT comes out as a special case.
- In the last decade, Gavril Farkas of Berlin has revolutionized our understanding of the space of curves of "intermediate" dimension, between ten (where the space ceases to be easilly seen to be unirational) to 24 (where it is known to be general type by work of Eisenbud, Harris, and Mazur). Farkas has turned his attention to the space of covers, the theory of Prym

varieties. Farkas explained his argument that this moduli space is of general type in genus at least 13. His argument is surprisingly classical in nature, by explicitly computing the class of a cleverly chosen effective divisor on the moduli space.

- Sam Payne, a Clay Research Fellow and one of the leaders in the theory of toric varieties, described applications of Frobenius splittings to the geometry of these varieties. Frobenius splittings originated as a positive characteristic construction, but Payne's work is characteristic-independent, and indeed he is motivated by classical questions about Schubert and toric varieties over the complex numbers. Suppose X is a projective toric variety, with an ample line bundle L and a nef line bundle L'. In the 1980's, Oda asked, if X is smooth, is the natural multiplication map $H^0(X, L) \otimes H^0(X, L') \to H^0(X, L \otimes L')$ surjective. For example, if the answer is affirmative in the special case where L' is a power of L, then the section ring $R(X, L) = \bigoplus_{m \geq 0} H^0(X, L^m)$ is normal. In the 1990's, Sturmfels conjecture that if X is smooth and $R(X, L)$ is normal then the ideal of relations $I(X, L)$ is generated in degree two. In 1995, Bøgvad, conjectured that if X is smooth then the section ring $R(X, L)$ is normal and Koszul. These conjectures are wide open, starting in dimension three. Nevertheless, a central problem in toric geometry is to understand under what hypotheses $R(X, L)$ is normal or Koszul. Payne explained how Frobenius splitting can give progress on such problems. For example, he showed that if X is diagonally split at some prime p and $L_1, \ldots, L_r$ are nef line bundles on X, then $R(X; L_1, \ldots, L_r) = \bigoplus_{a_1, \ldots, a_r} H^0(X, L_1^{a_1} \otimes \cdots \otimes L_r^{a_r})$ is normal and Koszul. In particular, the analogues of Oda's problem and both conjectures are true in this case, even if X is singular. In order to prove such results and apply them in practice, Payne proved a beautiful combinatorial characterization of diagonally split toric varieties. Using this criterion, he settles a longstanding question in the negative: There is a Schubert variety (of type G_2) that is not diagonally split.