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Progress in Surface Theory

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ABSTRACT. The theory of surfaces has undergone substantial changes in recent years, with many different active areas at this point in time. It has become mainstream to study minimal surfaces and constant mean curvature surfaces in more general target spaces than space forms. So far, the main strands of development center around methods from the theory of elliptic differential equations, complex analytic methods, and integrable systems techniques. The study of singular surfaces has widened the classical point of view and seems to be a particularly promising direction of research when considering certain moduli space problems.

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Introduction by the Organisers

The workshop *Progress in Surface Theory*, organised by Uwe Abresch (Bochum), Josef Dorfmeister (München), and Masaaki Umehara (Osaka) was held April 29th–May 5th, 2007. In recent years, Surface Theory has become a very active field of research again. Studying surfaces in more general ambient spaces than space forms and surfaces with certain types of mild singularities has become much more mainstream, even though a complete understanding of all important surface classes in space forms is still an issue. In order to facilitate these developments, it has been crucial to adopt methods from different fields. Methods from the theory of elliptic differential equations, complex analytic methods, and integrable systems techniques are now widely used.

It has been a major goal of this workshop to bring together many of the best experts in all these substantially different fields.

The workshop was attended by 26 participants from seven different countries, with particularly large contingents from Germany, Japan, France, and the US. This is quite an impressive list for a half-size workshop like this. On the other hand, every participant could be given the opportunity to present his/her work.

The official program consisted of 25 talks at 40–50 minutes each. In addition, two evenings were used for more informal presentations concerning the computer visualization of surfaces. Most lectures were well prepared and well presented and provided a lot of useful background information, facilitating the interaction between the different groups. It also provided an opportunity for young, in particular young German, mathematicians to have direct contact with a broad international group of established mathematicians and their work.

While the schedule was quite full, just for this reason the workshop indeed provided a fairly complete picture of the currently active areas of surface theory. Nevertheless, in the afternoons and the evenings there was still quite some time left for individual discussions, time that was extensively and intensively used by the participants. Several (even senior) participants commented to us that they had learned quite a bit from the talks and the interactions that they had with other participants. Certainly, the great setting and the superb atmosphere, for which Oberwolfach is renowned around the world, making it a unique conference place, provided the environment that generated many fruitful discussions.

We feel that the meeting was exciting and highly successful, in particular in view of some startling new results and the extraordinary amount of scientific interaction among the participants.

Here is a brief summary of the mathematical contents: one area of surface theory where a lot of progress has been made concerns qualitative estimates for minimal and constant mean curvature (cmc) surfaces in 3-manifolds with bounded geometry. W. H. Meeks has given a nice survey explaining the key techniques from analysis that these results are based on.

The move to more general ambient spaces than space forms got a big push when U. Abresch and H. Rosenberg discovered a natural holomorphic quadratic differential on cmc surfaces in a wide class of 3-dimensional homogeneous bundles. This discovery has widened the realm for explicit constructions (see, e.g., the generalized Lawson correspondences of B. Daniel).

L. Hauswirth presented interesting consequences of this: translating via the generalized Lawson correspondence minimal surfaces in the Heisenberg group $\text{Nil}(3)$ into cmc $1/2$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and vice versa. (As another high point in this development, I. Fernandez and P. Mira posted a paper shortly after the workshop, on arXiv, in which they solved the Bernstein problem in $\text{Nil}(3)$.) I. Taimanov and his student D. Berdinsky have been working on extending the integrable systems approach to cmc tori in this class of homogeneous bundles.

H. Rosenberg himself explained a particularly nice application of minimal surface theory in $\mathbb{H}^2 \times \mathbb{R}$ that he has made in collaboration with P. Collin. They have constructed a surjective *harmonic diffeomorphism* from the complex plane \mathbb{C} onto the hyperbolic plane \mathbb{H}^2 , thereby disproving a conjecture by S. T. Yau. By the

uniformization theorem, this diffeomorphism cannot be conformal. Moreover, it is known that there is no surjective harmonic diffeomorphism from the hyperbolic plane \mathbb{H}^2 onto the entire complex plane \mathbb{C} .

The *classical theory of minimal surfaces* in euclidean space is not a closed subject yet either. T. Shoda constructed irreducible immersed trigonal minimal surfaces in flat tori \mathbb{R}^n/Λ with $n > 3$ that cannot be immersed into some \mathbb{R}^3/Λ' . Moreover, generalizing earlier work of Nadirashvili, F. Martin has established a number of highly interesting *density results*. The most notable one states that there is an open dense set in the space of Jordan curves such that for each $\Gamma: \mathbb{S}^1 \rightarrow \mathbb{R}^3$ in this set there exists a complete minimal immersion $f: \mathbb{D} \looparrowright \mathbb{R}^3$ that extends continuously to the closed disc \mathbb{D} in such a way that the extension to the boundary $\partial\mathbb{D}$ coincides with Γ upon identifying $\partial\mathbb{D}$ and \mathbb{S}^1 by a suitable homomorphism. Moreover, f is not proper and the surface is hyperbolic.

The topic of cmc surfaces was represented in both its flavours, the qualitative theory and the integrable systems approach. The talks covering the *qualitative theory* centered around bifurcation and deformation arguments. J. Ratzkin presented a nice nondegeneracy result for the spectrum of properly embedded cmc surfaces of genus 0 in a solid slab in euclidean space. By a well-known theorem of Korevaar, Kusner, and Solomon the ends of such a surface are asymptotic to unduloids. The crucial step in Ratzkin's approach was to use an infinitesimal version of the Lawson correspondence in order to translate between even and almost odd Jacobi fields. A somewhat different, though closely related point of view on the deformation theory of Alexandrov embedded k -noids was presented by K. Große-Brauckmann, a talk which created a particular amount of follow-up investigations. F. Pacard on the other hand explained how one can foliate the ends of an asymptotically hyperbolic manifold by cmc hypersurfaces.

The k -noids were also a central theme in the work related to the *integrable systems approach* to cmc surfaces. J. Dorfmeister explained how all cmc trinoids of genus $g = 0$ can be constructed from basic complex potentials. M. Kilian presented a complementary result that he had established in joint work with W. Rossman and N. Schmitt: a holomorphic perturbation of the potential of an unduloid yields an annular cmc immersion that is C^∞ -asymptotic to the reference unduloid. In particular, he gave potentials describing k -noids with $k > 3$. There is hope that a combination of classical deformation theory and integrable systems methods will eventually lead to an understanding of the moduli spaces of k -noids.

The *discrete aspect* of the integrable systems approach was presented in the talk of W. Rossman. He described a 1-parameter family of discrete systems that can be used to study how the shape of asymptotically unduloidal ends of surfaces in \mathbb{H}^3 with constant mean curvature $H > 1$ changes when H approaches 1, and one enters the realm of Bryant surfaces.

S. Kobayashi on the other hand used the *inherently complex nature* of the integrable systems method in order to give a uniform description of many 2-dimensional 1-1 integrable systems as real forms of complex 2-dimensional constant mean curvature surfaces.

Integrable systems as a *unifying framework* also played a key role in the talks by D. Brander and C.L. Terng. D. Brander established a link between integrable systems based on 3-involution loop groups and geometric problems in certain classes of high-dimensional manifolds. In a similar spirit, C.L. Terng explained various applications of a fairly general class of systems, the U/K-systems, to surface theory.

F.E. Burstall, on the other hand, focussed on *specific integrable systems* modelled on 4-dimensional *4-symmetric spaces*, systems that are expected to tie in nicely with the concept of (generalized) Darboux transformations. The generalized Darboux transformations themselves were introduced by K. Leschke, and the Willmore surfaces were the central topic in the lecture by Ch. Bohle. In a similar context, F. Pedit discussed estimates for the Willmore functional for conformal tori of spectral genus 0.

Another extension of the classical approach to surface theory, represented prominently in this workshop, is the inclusion of singularities. Since, as is well known, singularities cannot be avoided in some cases, it is natural to consider a definition of “surfaces” which now admits singularities of certain types.

Cuspidal edges and swallowtails occur quite frequently as singularities. On the other hand, these two types of singularities are known to also occur generically in wave fronts. Therefore, a detailed investigation of the geometry of wave fronts is of great importance.

M. Umehara and M. Kokubu gave talks on flat surfaces in \mathbb{R}^3 and \mathbb{H}^3 as wave fronts, and presented several global results, including, the classification theorem of complete flat fronts in \mathbb{R}^3 and the orientability of flat surfaces in \mathbb{H}^3 . On the other hand, K. Yamada gave a talk on constant mean curvature 1 surfaces in de Sitter 3-space, which are the projection of null immersed curves in $\text{Sl}(2, \mathbb{C})$. He explained in a very clear and instructive talk that such surfaces admit cuspidal edges, swallowtails, and cuspidal cross caps in general, and illustrated his results by computer generated pictures.

It is important to note that the extension to “mildly singular” surfaces extends the class of classical surfaces considerably. It has the advantage of leading to more complete moduli spaces and will, as we expect, eventually lead to realistic classes of surfaces in general 3-dimensional manifolds, also reflecting many applications.

At the very end, we would like to mention the talk on flows on spaces of curves U. Pinkall. His talk intertwined classical surface theory, integrable systems methods, discrete surface theory and computer visualization and was particularly well received.

This particular workshop included two evenings with visualization presentations by R. Palais, *Virtual Math Museum*, F. Pedit, *Gallery of Surfaces*, and U. Pinkall, *JReality*.

(<http://VirtualMathMuseum.org>,

<http://www.mathematik.uni-tuebingen.de/ab/Differentialgeometrie/gallery/index.html>,

<http://www3.math.tu-berlin.de/jreality/>).