

MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

Report No. 19/2007

**Mini-Workshop: Geometric Measure Theoretic  
Approaches to Potentials on Fractals and Manifolds**

Organised by  
Peter Grabner (Graz)  
Douglas Hardin (Vanderbilt)  
Edward Saff (Vanderbilt)  
Martina Zähle (Jena)

April 8th – April 14th, 2007

ABSTRACT. The workshop brought together researchers and graduate students from different areas of mathematics, such as analysis, probability theory, geometry, and number theory. The topics of joint interest were motivated by recent problems in potential theory with impacts into these areas:

- discrete approximation to energy minimising measures
- potential theory on fractals and manifolds
- geometric measure theory on fractals
- probabilistic potential theory
- spectral theory on fractals and sets with fractal boundary.

The format of a mini-workshop was especially well-suited for our subject, since it allowed enough time for personal discussions besides the talks given by the participants.

The concept of energy of a charge distribution on a subset of Euclidean space is one of the core subjects of potential theory. Recent generalisations of this concept to hyper-singular energy kernels and discrete  $N$ -point distributions exhibit a close connection with ideas from geometric measure theory. A recent article by two of the organisers shows that  $N$ -point configurations minimising the discrete energy in the hyper-singular case can be used to characterise the Hausdorff measure on  $d$ -dimensional  $d$ -rectifiable manifolds embedded in Euclidean space. Such minimal energy point sets can be used for the discretisation of manifolds, which has numerous applications.

On the other hand discretisation by graph structures is a common means for analysis on fractal structures. Usually, a diffusion and an associated Laplace operator are defined by rescaling discrete random walks and their transition operators on the approximating graphs.

## Introduction by the Organisers

The mini-workshop *Geometric Measure Theoretic Approaches to Potentials on Fractals and Manifolds*, organised by Peter Grabner (Graz), Douglas Hardin (Vanderbilt), Edward B. Saff (Vanderbilt), and Martina Zähle (Jena) was held from April 8 to 14, 2007. The meeting had 17 participants from 6 countries. The participants had background from different areas such as fractal geometry, geometric measure theory, stochastic processes, and potential theory. This diversity gave rise to new interactions among the participants. In order to initiate these interactions and to put the focus on the main themes of the workshop, the first two days of the workshop were organised around three introductory lectures:

**Edward B. Saff:** An overview of discrete minimal energy problems on manifolds

**Martina Zähle:** Classical potential theory and stochastic processes

**Pertti Mattila:** Geometric measure theory on fractals.

Potential theory and geometric measure theory have many applications and interactions with various areas of mathematics. The workshop was focussed especially on applications in probability theory, fractal geometry, discrete minimum energy, and harmonic analysis.

Stochastic processes are a classical application of potential theory. Several talks during the workshop were devoted to this area. *Michael Hinz* discussed Dirichlet form techniques for the approximation of jump processes on fractal sets. In particular, he studied the influence of a weight function on the behaviour of the process. *Yimin Xiao* spoke on recent results on the behaviour of  $\alpha$ -stable Lévy processes. He discussed the connection between Lévy processes, energy forms, and the corresponding capacities. He suggested using these ideas for the study of fractal properties of more general Markov processes. *Martina Zähle* gave an introductory talk on the interplay between classical potential theory, stochastic processes, and their traces on fractals. She gave special emphasis to Riesz and Bessel potentials, as well as the corresponding function spaces. *Jiixin Hu* discussed Dirichlet forms on fractals and their domain. On post-critically finite (p. c. f.) fractals rescalings of finite difference operators are used to construct Dirichlet forms. Estimates for the effective resistance in terms of the distance were presented. *Zhenqing Chen* presented a new approach to the definition of reflecting Brownian motion on compact sets with non-smooth boundary. This is based on discrete approximation by random walks on finer and finer grids. It is shown that this definition is equivalent to other less constructive approaches.

Minimisation of discrete and continuous energies is a classical subject of potential theory. Recently, relations to questions originating from geometric measure theory arose, which were one of the motivations for this workshop. *Edward Saff* gave an introductory talk about recent joint work with S. Borodachov and D. Hardin on discrete minimal energy in the hyper-singular case. In contrast to classical potential theory for energy integrals, the discrete energy also exists for Riesz-kernels  $\|x - y\|^{-s}$  with  $s \geq d$  ( $d$  being the dimension of the set). Indeed,

it has been proved by D. Hardin and E. Saff that the configurations of minimal energy for such values of the Riesz parameter  $s$  are uniformly distributed with respect to normalised  $d$ -dimensional Hausdorff measure, as the number of points tends to  $\infty$ . *Douglas Hardin* presented the ideas of the proof, which is based on results for  $d$ -rectifiable sets, as well as self-similarity considerations. As  $s \rightarrow \infty$ , the minimal energy problem becomes the classical problem of best packing. This relates the determination of the asymptotic main term of minimal energy to packing problems. *Matthew Calef* discussed the limiting case  $s = d$ , which can be obtained from classical potential theory by taking limits  $s \rightarrow d-$  of suitably renormalised potentials. In this case the measure of minimal energy equals the normalised Hausdorff measure for compact subsets of  $\mathbb{R}^d$  with positive Lebesgue measure. *Johann Brauchart* discussed the support of the equilibrium measure on sets of revolution in  $\mathbb{R}^3$  for Riesz-potentials with  $0 < s < 1$ . In particular, he discussed the question of, when the support is a proper subset of the outer boundary of the surface of revolution. *Abey Lopez* presented asymptotic distribution results for Leja points on the circle. He showed that these points form a uniformly distributed sequence, although their Riesz- $s$ -energy is asymptotically strictly larger than the minimal energy (for  $s > 1$ ).

Geometric measure theory and potential theory provide important techniques in the investigation of properties of fractal sets. *Pertti Mattila* presented methods and results from the geometric measure theory of fractals. Special emphasis was given to projection properties of fractal sets, such as the Hausdorff dimension of the “generic” projection of a set of given dimension. *Daniel Mauldin* discussed constructions of fractal sets based on (possibly infinite) iterated function systems of conformal maps in  $\mathbb{R}^n$ . He described tools and techniques which have been developed to analyse the properties of the limit set: Hausdorff, packing, Minkowski, or packing dimensions, as well as the quantisation dimension of Gibbs states and equilibrium measures of various potentials.

Classical potential theory Riesz, Bessel, and more general kernels and positive harmonic functions were the subject of three talks. *Volodymyr Andriyevskyy* gave a talk on positive harmonic functions in  $\mathbb{C} \setminus E$  depending on geometric properties of the set  $E \subset \mathbb{R}$ . He discussed necessary and sufficient conditions for the existence of two linearly independent positive harmonic functions. *Natalia Zorii* presented results on the existence of equilibrium measures on non-compact subsets of  $\mathbb{R}^n$ . *Peter Dragnev* discussed the supports of equilibrium on the sphere under the presence of a Riesz external field. He gave applications of his results to separation of discrete minimal energy point configurations.

Spectral theory on fractal set, its relation to complex dynamics, and corresponding zeta functions were the subject of two talks. *Michel Lapidus* gave an introduction to the theory that he and his coauthors developed of complex dimensions of fractal sets. Relations to tube formulæ of self-similar fractals, Minkowski measurability, and spectral asymptotics on sets with fractal boundary were discussed. *Peter Grabner* presented results on the analytic continuation of the spectral zeta

function on certain self-similar fractals and the relation to the classical Poincaré functional equation from complex dynamics.