

Report No. 9/2007

Normal Families and Complex Dynamics

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February 18th – February 24th, 2007

ABSTRACT. The schedule comprised more than 25 ordinary and problem session talks from a broad range of areas in function theory, including but not limited to: Nevanlinna theory, iteration of rational functions, dynamics of transcendental entire and meromorphic functions, function algebras, Riemann surfaces, each of them in close connection to the main topic Normal Families and Complex Dynamics.

Mathematics Subject Classification (2000): 30D05, 30D35, 30D45, 30F10, 37F10, 37F15, 37F45.

Introduction by the Organisers

The workshop *Normal Families and Complex Dynamics*, organised by Phil Rippon (Milton Keynes), Norbert Steinmetz (Dortmund) and Lawrence Zalcman (Ramat-Gan) was held February 18th–February 24th, 2007.

Normal families and Nevanlinna theory. Almost half a century after it first appeared, Hayman's seminal work¹ continues to exert a pervasive influence on the study of the relation between properties enjoyed by meromorphic functions on the complex plane and the normality of families of meromorphic functions on plane domains.

If $a, b \in \mathbb{C}$, $a \neq 0$, and f is a transcendental meromorphic function, then $f + a(f')^n - b$ vanishes infinitely often for $n \geq 3$. In his talk *Mingliang Fang* proved that this remains true for $n = 2$, thus answering an old question of Ye. He also showed that if f is a meromorphic function on \mathbb{C} , which has all but finitely many poles multiple and shares the distinct values $0, a, b$ (counting multiplicity) with f' , then $f \equiv f'$ unless a and b are related in a very specific way. *Jürgen Grahl* considers the condition $\psi(z) \equiv f^n(z) + af^{(k)}(xz) - b \neq 0$, where $a, b \in \mathbb{C}$,

¹*Picard values of meromorphic functions and their derivatives*, Ann. Math. **70** (1959), 9–42.

$a \neq 0$, and $0 < |x| \leq 1$. It turns out that for a family \mathcal{F} of holomorphic functions on the unit disc \mathbb{D} , this condition implies that \mathcal{F} is normal at 0 if $n \geq 3$ and $k \geq 1$. For meromorphic functions on \mathbb{C} , this same condition implies that f is constant for $n \geq k + 5$. *Shahar Nevo's* contribution focuses on results just beyond the range of application of normal family techniques. For instance, the family \mathcal{F} of meromorphic functions on a plane domain D , all of whose zeros have multiplicity at least $k + 1$ and such that $f^{(k)}(z) \neq 1$ for each $f \in \mathcal{F}$ and $z \in D$, is not normal; but it is quasiregular of order 1. As a consequence, one can show that the derivative of a transcendental meromorphic function which has at most finitely many simple zeros takes on every nonzero value infinitely often.

Escaping sets. The complex dynamics talks focused on two main topics, one being properties of the escaping set. The early work on the escaping set of a general transcendental entire function was done by Eremenko, who established the close connection between the escaping set and the Julia set. He asked two questions that have provoked great interest: *are all components of the escaping set unbounded and, more strongly, are these components path connected to infinity?*

Walter Bergweiler described a generalisation of Wiman-Valiron theory to a meromorphic function with a direct tract. Using this, one can show that many properties of the escaping set that are true for an entire function but false for a general meromorphic function (for example, that this set has at least one unbounded component) remain true for a meromorphic function with a direct tract. *Dierk Schleicher* described an example of an entire function of bounded type (that is, one in the Eremenko-Lyubich class) for which every path component of the escaping set is bounded, thus showing that the answer to the strong form of Eremenko's question is 'no'. However, for an entire function of bounded type and finite order the escaping set is a disjoint union of curves, so in this case the answer is 'yes'. *Lasse Rempe* discussed an analog of Böttcher's theorem (that two polynomials of the same degree are conformally conjugate near infinity) for bounded type entire functions. If two such functions are quasiconformally equivalent near infinity, then they are quasiconformally conjugate on a set related to the escaping set, and this explains why their Julia sets look similar near infinity. *Bogusława Karpińska* showed how coding trees can be used to describe the dynamics on the Julia set of an entire function whose inverse function singularities are compactly contained in an immediate basin of attraction, in particular discussing which points of the Julia set are accessible from within the basin. *Gwyneth Stallard* spoke about sufficient conditions for the escaping set of an entire function to be connected, giving a number of topological conditions which guarantee that this is true. From these conditions, and earlier work proving the absence of unbounded Fatou components, it can be deduced that the answer to Eremenko's question is 'yes' for many functions of small growth.

Connectivity of Fatou components. The second main topic in the complex dynamics part focused on connectivity properties of Fatou components.

Phil Rippon gave conditions for a meromorphic function with a finite number of poles to have a Baker wandering domain, and deduced that the eventual connectivity of a wandering domain of such a function is either two or infinity, in the case of a Baker wandering domain, and one otherwise. *Patricia Domínguez* showed how to use quasi-conformal surgery to construct examples of various types of functions (e.g. meromorphic functions and functions meromorphic outside small sets) with doubly-connected wandering domains in various configurations. *Marcus Stiemer* considered a class of generalized Blaschke products that can have zeros outside the unit disc as well as inside, and showed how to construct examples of functions in this class which have Fatou components of any given connectivity number. *Richard Stankewitz* discussed the dynamics of polynomial semigroups with bounded postcritical set and described some of their properties; for example, if the Fatou set has two doubly connected components, then the Julia set contains a Cantor family of quasicircles. *Nuria Fagella* discussed work on a generalization of a result of Shishikura, showing that if a meromorphic function has a multiply connected Fatou component which is of a certain type, then the function must have a weakly repelling fixed point.

Hausdorff dimension and measure. *Ludwik Jaksztas* described work related to the limiting behaviour of the Hausdorff dimension of certain quadratic Julia sets near the parameter values $1/4$ and $-3/4$. *Jörn Peter* extended a result of McMullen by showing that for certain exponential functions with an attracting fixed point there is a gauge function, related to the Schröder function of the fixed point, with respect to which the Hausdorff measure of the Julia set is infinity.

Miscellanea. *Dzmitry Dudko* described a proof, due to Adam Epstein, of the Fatou-Shishikura inequality. *Lukas Geyer* presented a survey on the type problem, which consists in deciding whether a Riemann surface (X, f) spread over the plane is hyperbolic or parabolic. He showed that (X, f) is type-stable if it has uniformly separated singularities, but showed also that there exists a parabolic type-stable surface (X, f) whose singularities are not uniformly separated. *Aimo Hinkkanen's* talk focussed on majorants of analytic functions, i.e. the question whether or not conditions imposed on analytic functions on the boundary of their domain of definition remain valid in the interior. A typical question is to ask whether $|f(z_1) - f(z_2)| \leq \mu(|z_1 - z_2|)$ with $z_1 \in \partial G$ fixed and $z_2 \in \partial G$ implies $|f(z_1) - f(z_2)| \leq C\mu(|z_1 - z_2|)$ for $z_2 \in G$, C some constant. Under reasonable circumstances this was proved with C independent of μ . *James Langley* talked about value distribution properties of functions in the classes S and B (functions meromorphic in \mathbb{C} having finitely many resp. a bounded set of singularities of f^{-1}). In particular he proved the conjectures of Mues for functions in B having finite lower order, and Gol'dberg's conjecture for functions in S , without any further restriction. *Oliver Roth* proved existence of a Blaschke product with prescribed sequence (z_ν) of critical points (satisfying the Blaschke condition); existence was known before only in the case of finitely many points. Roth's proof is based on solvability of the elliptic boundary value problem $\Delta u = 4|h(z)|^2 e^{2u}$ on \mathbb{D} , $u \rightarrow \infty$ as $z \rightarrow \partial\mathbb{D}$, with h the Blaschke product with zeros z_ν . *Nikita Selinger* gave an

interesting description of Douady's and Hubbard's proof of Thurston's theorem, which characterizes when a postcritically finite branched cover of the Riemann sphere is equivalent to a rational map.