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Calculus of Variations

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ABSTRACT. Research in the Calculus of Variations has always been motivated by questions generated within the field itself as well as by problems arising in other areas of Mathematics, and applied sciences. The purpose of this meeting was to illustrate the diversity of points of view and motivations in current research, with particular attention to applications.

Mathematics Subject Classification (2000): 26B25, 35B65, 35J60, 35J20, 35K55, 49J45, 49L25, 49Q10, 49Q20, 49R50, 52A40, 53C42, 58E20, 83C30, 94A17.

Introduction by the Organisers

The workshop “Calculus of Variations” took place from July 9 to 15, 2006, and was attended by almost fifty participants, mostly from European and North American universities and research institutes. There were 24 lectures on recent research topics, plus a review lecture on the Lieb-Thirring inequalities by Michael Loss (Georgia Tech, Atlanta). As the workshop had no specific focus, talks covered a wide range of topics, with the aim of featuring different research trends, bringing new problems to the fore, and stimulating interaction between mathematicians from different backgrounds.

Five lectures were focused on problems related to Continuum Mechanics and Materials Science. Gero Friesecke (Munich and Warwick) presented some results on a simplified model for molecules, where the aim is to give a rigorous explanation of the screening effect (i.e., the fact that the interaction of electrically balanced molecules due to electrostatic forces is short ranged); this problem is still open and presumably quite challenging in case of ‘realistic’ models. László Székelyhidy (ETH Zürich) presented new result about the structure of quasicontvex hulls for sets of 2×2 matrices, perhaps the most interesting development on this topic in

recent years. Sergio Conti (Duisburg) considered the asymptotic behaviour of an energy functional that appears in the modeling of different physical problems, such as blistering in elastic films, magnetic thin films, etc.; the main result presented in his lecture adds one more piece to the work of many authors towards the proof of a conjecture by Aviles and Giga on the variational limit of such functional. The lecture by Felix Otto (Bonn) was focused on the rigorous analysis of pattern formation in micromagnetics: this type of pattern formation is particularly interesting because of the complexity of the observed behaviours – not yet fully explained in rigorous terms – and of the relative simplicity of the underlying continuum model. Related to this topic was also the lecture of Hans Knüpfer (Bonn).

Four lectures dealt with regularity problems of different sorts. G. Rosario Mingione (Parma) reviewed some recent developments on the regularity of solutions of nonlinear parabolic systems. Michael Struwe (ETH Zürich) presented a new approach to regularity for harmonic maps valued in a hypersurfaces, yielding new results when the domain dimension is larger than 2. The regularity of harmonic maps valued in Riemannian manifolds was also considered by Ernst Kuwert (Freiburg); these results stemmed from other results on the conformal structure of surfaces with suitable bounds on the Willmore energy. Mariel Saez (MPI for Gravitational Physics, Potsdam) presented a Lipschitz regularity result for the pseudo-infinity Laplacian.

A certain number of lectures were related to shape optimization and optimal transport problems. Almut Burchard (Toronto) presented some partial results about the shape of closed curves in the three-dimensional space that minimize the first eigenvalue of the associated one-dimensional Schrödinger operator; it is conjectured that these curves are circles (among other things, the conjecture is related to the optimal constant in a particular Lieb-Thirring inequality). Jochen Denzler (Knoxville) and Giuseppe Buttazzo (Pisa) considered other optimization problems related to the first eigenvalue of (variants of) the Laplace operator on a given domain. Alexander Plakhov (Aveiro) studied bodies of minimal resistance moving through a rarefied particle gas. Francesco Maggi (Duisburg) and Aldo Pratelli (Pavia) presented some recent quantitative versions with optimal exponents of the classical isoperimetric inequality in the n -dimensional Euclidean space. Qinglan Xia (UC Davis) proposed a model for the shape formation in tree leaves which postulates a step-by-step optimized growth for the associated transport system (the venation of the leaf), where “optimized” refers to a given transport cost. Numerical simulations based on this simple model show that varying the two built-in parameters generates a wide variety of leaf shapes. Vladimir Oliker (Emory University, Atlanta) described a variational approach to the Aleksandrov problem about the existence of closed convex hypersurfaces with prescribed integral Gauss curvature. A similar approach is also used to design reflecting surfaces with prescribed irradiance properties; the functional underlying this variational principle is related to Monge-Kantorovich optimal transport theory.

Yann Brenier (Nice) considered the problem of foliating the three-dimensional Euclidean space and the four-dimensional Minkowski space by extremal surfaces

(which in Minkowski space can be interpreted as classical relativistic strings). One way of obtaining such foliations is finding minimizers or critical points of suitable energy functionals, subject to certain nonlinear constraints; due to these constraints, standard methods do not apply in this case, and the existence of such minimizers is open. Pierre Cardaliaguet (Brest) studied a non-local geometric evolution problem for sets in the n -dimensional Euclidean space, which can be formally viewed as the gradient flow of a linear combination of volume and capacity. Since this flow preserves inclusion, it allows for a notion of weak solutions in the sense of viscosity; it is shown that such solutions agree with the limits of the the minimizing movements obtained by time discretization.

Diogo Gomes (Instituto Superior Tecnico, Lisbon) reviewed some recent results on the viscosity solution of Hamilton-Jacobi equations and the relations with the associated Hamiltonian dynamics, and Aubrey-Mather theory. Olivier Druet (ENS Lyon) presented new results on the bubbling phenomenon for the solutions (and also the Palais-Smale sequences) of sequences of variational elliptic equations in dimension two with critical nonlinearities. Robert Jerrard (Toronto) described a version of the Γ -convergence method designed for saddle points instead of minima, and used this abstract tool to obtain non-trivial solutions to the Ginzburg-Landau system in dimension three. Reiner Schätzle (Tübingen) gave a proof of (a modified version of) a conjecture by De Giorgi on the approximation of the Willmore functional for hypersurfaces in dimension three; the conjecture is still open in higher dimensions.

Keith Ball (University College London) presented the proof of a long-standing conjecture (due to Lieb) on the entropy gap between the normalized sum of N independent copies of a given random variable X and its limit as $N \rightarrow \infty$, i.e., the Gaussian distribution. A key role in the proof is played by a new variational characterization of Fisher information.

The lecture by Gerhard Huisken (MPI for Gravitational Physics, Potsdam) was focused on the problem of defining mass in general relativity; in particular, he presented a new definition based on the isoperimetric inequality (more precisely, on the asymptotic behaviour of the isoperimetric profile), and some results on the properties of this mass. One of the advantages of this definition, compared to others based on the notion of curvature, is the relatively simple calculus that is required for handling it. Furthermore, it can be adapted so as to obtain a notion of localized mass.

