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**Arbeitsgemeinschaft mit aktuellem Thema:
Modern Foundations for Stable Homotopy Theory**

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ABSTRACT. In recent years, *spectral algebra* or *stable homotopical algebra* over structured ring spectra has become an important new direction in stable homotopy theory. This workshop provided an introduction to structured ring spectra and applications of spectral algebra, both within homotopy theory and in other areas of mathematics.

Mathematics Subject Classification (2000): 55Pxx.

Introduction by the Organisers

Stable homotopy theory started out as the study of generalized cohomology theories for topological spaces, in the incarnation of the stable homotopy category of spectra. In recent years, an important new direction became the *spectral algebra* or *stable homotopical algebra* over structured ring spectra. Homotopy theorists have come up with a whole new world of ‘rings’ which are invisible to the eyes of algebraists, since they cannot be defined or constructed without the use of topology; indeed, in these ‘rings’, the laws of associativity, commutativity or distributivity only hold up to an infinite sequence of coherence relations. The initial ‘ring’ is no longer the ring of integers, but the *sphere spectrum* of algebraic topology; the ‘modules’ over the sphere spectrum define the stable homotopy category. Although ring spectra go beyond algebra, the classical algebraic world is properly contained in stable homotopical algebra. Indeed, via Eilenberg-Mac Lane spectra, classical algebra embeds into stable homotopy theory, and ordinary rings form a full subcategory of the homotopy category of ring spectra. Topology interpolates algebra in various ways, and when rationalized, stable homotopy theory tends to become purely algebraic, but integrally it contains interesting torsion information.

There are plenty of applications of structured ring spectra and spectral algebra within homotopy theory, and in recent years, these concepts have started to

appear in other areas of mathematics (we will sketch connections to algebraic K -theory and arithmetic, and to algebraic geometry via motivic homotopy theory and derived algebraic geometry).

After this general introduction, we now want to give some history and more background and point out some areas where structured ring spectrum methods have been used for calculations, theorems or constructions. Several of the following topics were taken up in the talks of this AG.

Some history. A crucial prerequisite for spectral algebra is an associative and commutative smash product on a good point-set level category of spectra, which lifts the well-known smash product pairing on the *homotopy category*. To illustrate the drastic simplification that occurred in the foundations in the mid-90s, let us draw an analogy with the algebraic context. Let R be a commutative ring and imagine for a moment that the notion of a chain complex (of R -modules) has not been discovered, but nevertheless various complicated constructions of the unbounded derived category $\mathcal{D}(R)$ of the ring R exist. Moreover, constructions of the *derived* tensor product on the *derived* category exist, but they are complicated and the proof that the derived tensor product is associative and commutative occupies 30 pages. In this situation, you could talk about objects A in the derived category together with morphisms $A \otimes_R^L A \rightarrow A$, in the derived category, which are associative and unital, and possibly commutative, again in the derived category. This notion may be useful for some purposes, but it suffers from many defects – as one example, the category of modules (under derived tensor product in the derived category), does not in general form a triangulated category.

Now imagine that someone proposes the definition of a chain complex of R -modules and shows that by formally inverting the quasi-isomorphisms, one can construct the derived category. She also defines the tensor product of chain complexes and proves that tensoring with a bounded below (in homological terms) complex of projective modules preserves quasi-isomorphisms. It immediately follows that the tensor product descends to an associative and commutative product on the derived category. What is even better, now one can suddenly consider differential graded algebras, a ‘rigidified’ version of the crude multiplication ‘up-to-chain homotopy’. We would quickly discover that this notion is much more powerful and that differential graded algebras arise all over the place (while chain complexes with a multiplication which is merely associative up to chain homotopy seldom come up in nature).

Fortunately, this is not the historical course of development in homological algebra, but the development in stable homotopy theory was, in several aspects, as indicated above. The first construction of what is now called ‘the stable homotopy category’, including its symmetric monoidal smash product, is due to Boardman (unpublished); accounts of Boardman’s construction appear in [57] and [1, Part III] (Adams has to devote more than 30 pages to the construction and formal properties of the smash product). With this category, one could consider ring spectra ‘up to homotopy’, which are closely related to multiplicative cohomology theories.

However, the need and usefulness of ring spectra with rigidified multiplications soon became apparent, and topologists developed different ways of dealing with them. One line of approach used operads for the bookkeeping of the homotopies which encode all higher forms of associativity and commutativity, and this led to the notions of A_∞ - respectively E_∞ -ring spectra. Various notions of point-set level ring spectra had been used (which were only later recognized as the monoids in a symmetric monoidal model category). For example, the orthogonal ring spectra had appeared as \mathcal{S}_* -prefunctors in [34], the *functors with smash product* were introduced in [6] and symmetric ring spectra appeared as *strictly associative ring spectra* in [18, Def. 6.1] or as *FSPs defined on spheres* in [19, 2.7].

At this point it had become clear that many technicalities could be avoided if one had a smash product on a good point-set category of spectra which was associative and unital *before* passage to the homotopy category. For a long time no such category was known, and there was even evidence that it might not exist [28]. In retrospect, the modern spectra categories could maybe have been found earlier if Quillen's formalism of *model categories* [37, 23] had been taken more seriously; from the model category perspective, one should not expect an intrinsically 'left adjoint' construction like a smash product to have good homotopical behavior in general, and along with the search for a smash product, one should look for a compatible notion of cofibrations.

In the mid-90s, several categories of spectra with nice smash products were discovered, and simultaneously, model categories experienced a major renaissance. Around 1993, Elmendorf, Kriz, Mandell and May introduced the *S-modules* [14] and Jeff Smith gave the first talks about *symmetric spectra*; the details of the model structure were later worked out and written up by Hovey, Shipley and Smith [24]. In 1995, Lydakis [29] independently discovered and studied the smash product for Γ -spaces (in the sense of Segal [51]), and a little later he developed model structures and smash product for *simplicial functors* [30]. Except for the *S-modules* of Elmendorf, Kriz, Mandell and May, all other known models for spectra with nice smash product have a very similar flavor; they all arise as categories of continuous, space-valued functors from a symmetric monoidal indexing category, and the smash product is a convolution product (defined as a left Kan extension), which had much earlier been studied by category theorist Day [9]. This unifying context was made explicit by Mandell, May, Schwede and Shipley in [33], where another example, the *orthogonal spectra* was first worked out in detail. The different approaches to spectra categories with smash product have been generalized and adapted to equivariant homotopy theory [11, 31, 32] and motivic homotopy theory [12, 25, 26]. In this AG we will present these various setups with an emphasis on symmetric spectra.

Algebra versus homotopy theory. Many constructions and invariants for classical rings have counterparts for structured ring spectra. These ring spectra have well behaved module categories; algebraic K -theory, Hochschild homology and André-Quillen homology admit refinements; classical constructions such as localization, group rings, matrix rings, Morita theory and Galois theory carry over,

suitably adapted. This export of concepts from algebra to topology illuminates both fields. For example, Dwyer, Greenlees, and Iyengar [13] have shown that Gorenstein duality, Poincaré duality, and Gross-Hopkins duality become results of a single point of view. Morita theory for ring spectra [4, 50, 48] gives a new perspective at *tilting theory*. The extension of Galois theory to ring spectra [42] has genuinely new kinds of examples given by classical and higher forms of real topological K -theory. Certain algebraic extensions of commutative rings can be lifted to the sphere spectrum; for example, roots of unity can be adjoined to E_∞ -ring spectra away from ramification [46]. Some algebraic notions, for example the units or the center of a ring, are more subtle, and their generalizations to ring spectra show richer features than the classical counterparts.

Power operations. Recognizing a multiplicative cohomology theory as an E_∞ -ring spectrum leads to additional structure which can be a powerful theoretical and calculational tool. We want to illustrate this by the example of the Adams spectral sequence [1, III.15], [35, II.9], [52, Ch. 19], a tool which has been used extensively for calculations of stable homotopy groups. In its most classical instance, the Adams spectral sequence converges to the p -completed stable homotopy groups of spheres and takes the form

$$E_2^{s,t} = \text{Ext}_{\mathcal{A}_p}^{s,t}(\mathbb{F}_p, \mathbb{F}_p) \implies (\pi_{t-s}^{\text{stable}} S^0)_p^\wedge .$$

The E_2 -term is given by Ext-groups over the Steenrod algebra \mathcal{A}_p of stable mod- p cohomology operations. This spectral sequence neatly separates the problem of calculating homotopy groups into an algebraic and a purely homotopy theoretic part.

The Steenrod algebra has various explicit descriptions and the E_2 -term can be calculated mechanically. In fact, computer calculations have been pushed up to the range $t - s \leq 210$ (for $p = 2$) [8, 36]. The 1-line is given by the primitive elements in the dual Steenrod algebra, which for $p = 2$ are classes $h_i \in \text{Ext}_{\mathcal{A}_2}^{1,2^i}(\mathbb{F}_2, \mathbb{F}_2)$ for $i \geq 0$. The first few classes h_0, h_1, h_2 and h_3 are infinite cycles and they detect the Hopf maps $2\iota, \eta, \nu$ and σ ; the Hopf maps arise from the division algebra structures on \mathbb{R}, \mathbb{C} , the quaternions and the Cayley numbers.

On the other hand, identifying differentials and extensions in the Adams spectral sequence is a matter of stable homotopy theory. The known differentials can be derived by exploiting more and more subtle aspects of the homotopy-commutativity of the stable homotopy groups of spheres (or, in our jargon, the E_∞ -structure of the sphere spectrum). The first d_2 -differential in the mod-2 Adams spectral sequence is a consequence of the graded-commutativity $yx = (-1)^{|x||y|}xy$ in the ring of stable homotopy groups of spheres: like any element of odd dimension, the third Hopf map σ in the stable 7-stem has to satisfy $2\sigma^2 = 0$. The class $h_0h_3^2$ which represents $2\sigma^2$ in $E_2^{3,17}$ is non-zero, so it has to be in the image of some differential. In these dimensions, the only possible differential is $d_2(h_4) = h_0h_3^2$, which simultaneously proves that the class $h_4 \in E_2^{1,16}$ is not an infinite cycle and thus excludes the existence of a 16-dimensional real division algebra.

Graded-commutativity of the multiplications is only a faint shadow of the E_∞ -structure on the sphere spectrum. A more detailed investigation reveals that an E_∞ -structure on a ring spectrum yields *power operations* [7, Ch. I], [40, Sec. 7]. Various kinds of power operations can be constructed on the homotopy and (generalized) homology groups of the E_∞ -ring spectrum and on the E_2 -term of the Adams spectral sequence. These operations interact in certain ways with the differential, and this interaction is an effective tool for determining such differentials. For example, the operations propagate the first d_2 -differential, which proves that the classes h_i for $i = 0, 1, 2$ and 3 are the only infinite cycles on the 1-line and excludes the existence of other real division algebras.

Algebraic K -theory. Structured ring spectra and algebraic K -theory are closely related in many ways, and in both directions. Algebraic K -theory started out as the Grothendieck group of finitely generated projective modules over a ring. Quillen introduced the higher algebraic K -groups as the homotopy groups of a certain topological space, for which he gave two different constructions (the *plus-construction* and the *Q -construction* [38]). Nowadays, algebraic K -theory can accept various sorts of categorical data as input, and produces spectra as output (the spaces Quillen constructed are the underlying infinite loop spaces of these K -theory spectra). For example, the symmetric monoidal category (under direct sum) of finitely generated projective modules leads to the K -theory spectrum of a ring, and the category of finite sets (under disjoint union) produces the sphere spectrum. If the input category has a second symmetric monoidal product which suitably distributes over the ‘sum’, then algebraic K -theory produces commutative ring spectra as output.

It was Waldhausen who pioneered the algebraic K -theory of structured ring spectra, introduced the *algebraic K -theory of a topological space X* as the algebraic K -theory of the *spherical group ring* $S[\Omega X] = \Sigma^\infty \Omega X_+$ and established close links to geometric topology, see [58]. Waldhausen could deal with such ring spectra, or their modules, in an unstable, less technical fashion before the modern categories of spectra were found: his first definition of the algebraic K -theory of a ring spectrum mimics Quillen’s plus-construction applied to the classifying space of the infinite general linear group. Similarly, Bökstedt [6] (in cooperation with Waldhausen) defined *topological Hochschild homology* using so-called *Functors with Smash Product*, which were only later recognized as the monoids with respect to the smash product of simplicial functors [30]. In any event, a good bit of the development of A_∞ -ring spectra was motivated by the applications in algebraic K -theory.

Later, Waldhausen developed one of the most flexible and powerful K -theory machines, the *S_\bullet -construction* [59] which starts with a *category with cofibrations and weak equivalences* (nowadays often referred to as a *Waldhausen category*). Originally, Waldhausen associated to this data a sequential spectrum and indicated [59, p. 342] how a categorical pairing leads to a smash product pairing on K -theory spectra. The S_\bullet -construction can be applied to a category of cofibrant

highly structured module spectra. Applying this to Eilenberg-Mac Lane ring spectra gives back Quillen K -theory, and applied to spherical group rings of the form $S[\Omega X]$ gives an interpretation of Waldhausen's $A(X)$.

Hesselholt eventually recognized that the iterated S_\bullet -construction gives K -theory as a symmetric spectrum and that it turns good products on the input category into symmetric ring spectra [16, Prop. 6.1.1]. Elmendorf and Mandell [15] have a different way of rigidifying the algebraic K -theory of a bipermutative category into an E_∞ -symmetric ring spectrum.

Units of a ring spectrum. One of the more subtle generalizations of a classical construction is that of the units of a ring spectrum. The units of a structured ring spectrum form a loop space, and in the presence of enough commutativity (i.e., for E_∞ - or strictly commutative ring spectra), the units even form an infinite loop space. These observations are due to May and are highly relevant to orientation theory and bordism. A recent application of these techniques is the construction by Ando, Hopkins and Rezk, of E_∞ -maps $MO\langle 8 \rangle \rightarrow \mathbf{tmf}$ from the string-cobordism spectrum to the spectrum of topological modular forms which realize the Witten genus on homotopy groups [21, Sec. 6] (this refines earlier work of Ando, Hopkins and Strickland on the $MU\langle 6 \rangle$ -orientation of elliptic spectra [20, 2, 3]). The details of this are not yet publicly available, but see [21, Sec. 6], [27], [40].

Waldhausen's first definition of algebraic K -theory of a ring spectrum is based on Quillen's plus-construction and uses the units of matrix ring spectra. While for a commutative ring in the classical sense, the units are always a direct summand in the first K -group, Waldhausen showed that the units of the sphere spectrum do not split off its K -theory spectrum, not even on the level of homotopy groups. This phenomenon has been studied systematically by Schlichtkrull [45], producing non-trivial classes in the K -theory of an E_∞ -ring spectrum from homotopy classes which are not annihilated by the Hopf map η .

Homotopical algebraic geometry. In this Arbeitsgemeinschaft, we are trying to convey the idea that the foundations of multiplicative stable homotopy theory are now in good shape, and ready to use. In fact, the machinery allows to 'glue' commutative ring spectra into more general algebro-geometric objects, making them the affine pieces of 'schemes' or even 'stacks'. This area is becoming known as *homotopical algebraic geometry*, and one set of foundations has been pioneered by Toën and Vezzosi in a series of papers [53, 54, 55, 56].

Another promising line of research is to investigate small (e.g., Deligne-Mumford) stacks which come equipped with a flat morphism to the moduli stack of formal groups; one might hope to lift the graded structure sheaf of such a stack to a sheaf of ring spectra and capture a snapshot of stable homotopy theory. In this context, one may think of the generalized algebro-geometric objects as an ordinary scheme or stack, together with a sheaf of E_∞ -ring spectra, which locally looks like 'Spec' of an E_∞ -ring spectrum. The structure sheaf of the underlying ordinary stack can be recovered by taking π_0 of the sheaf of ring spectra. It is essentially by this program that Hopkins and his coworkers produced *topological modular forms* (the 'universal' version of elliptic cohomology) and the recent work of Lurie

shows that these ideas can be extended to almost any situation where the Serre-Tate theorem on deformations applies. Here the flow of information goes both ways: the number theory informs homotopy theoretic calculations, but surprising algebraic phenomena, such as the Borcherds congruence in modular forms, have natural homotopy theoretic explanations [21, Thm. 5.10].

While this is a ‘hot’ area in algebraic topology, the organizers decided not to have talks about it in this AG because of the lack of publicly available literature (but see [5, 17, 20, 21, 22, 27, 39]).

Rigidity theorem. After having discussed various models for the stable homotopy category, the last day of the AG will be devoted to its ‘rigidity’ property [49]; this says that the stable homotopy category admits essentially only one model. More precisely, any model category whose homotopy category is equivalent, as a triangulated category, to the homotopy category of spectra is already *Quillen equivalent* to the model category of spectra. Loosely speaking, this says that all higher order homotopy theory is determined by the homotopy category, a property which is very special. Examples of triangulated categories which have inequivalent models are given in [47, 2.1, 2.2] or [10, Rem. 6.8] (which is based on [44]).

In algebra, a ‘rigidity theorem’ for unbounded derived categories of rings is provided by *tilting theory*; it is usually stated in the form that if two rings are derived equivalent, then, under a flatness assumption, there is a complex of bimodules X such that derived tensor product with X is an equivalence of triangulated categories [41] (a reworking of this result in model category terms, which also removes the flatness assumption, can be found in [10, Thm. 4.2]). Rigidity fails for categories of dg-modules over differential graded algebras [10, Rem. 6.8]. Incidentally, neither for derived categories of rings nor for the stable homotopy category is it known whether every derived equivalence lifts to a Quillen equivalence, or equivalently, whether there are exotic self-equivalences of these triangulated categories.

Questions about rigidity and exotic models are related to the problem of whether algebraic K -theory is an invariant of triangulated categories. Quillen equivalent model categories have equivalent K -theory spectra [10, Cor. 3.10], [43], and in special ‘rigid’ situations, the triangulated category determines the model, and thus the algebraic K -theory.

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