Mathematisches Forschungsinstitut Oberwolfach

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Arbeitsgemeinschaft mit aktuellem Thema: Algebraic Cobordism

Organised by
Marc Levine (Boston)
Fabien Morel (München)

April 4th – April 8th, 2005

Abstract. The aim of this Arbeitsgemeinschaft was to present the theory of Algebraic Cobordism due to Marc Levine and Fabien Morel through the lines of their original articles:
Inspired by the work of Quillen on complex cobordism, one first introduces the notion of oriented cohomology theory on the category of smooth varieties over a field k. Grothendieck's method allows one to extend the theory of Chern classes to such theories. When $char(k) = 0$, one proves the existence of a universal oriented cohomology theory $X \to \Omega^*(X)$. Localisation and homotopy invariance are then proved for this universal theory. For any field k of characteristic 0 one can prove for algebraic cobordism the analogue of a theorem of Quillen on complex cobordism: the cobordism ring of the ground field is the Lazard ring $\mathbb{L}$ and for any smooth k-variety X, the algebraic cobordism ring $\Omega^*(X)$ is generated, as an $\mathbb{L}$-module, by elements of non negative degree. This implies Rost's conjectured degree formula. One also gives a relation between the Chow ring, the K_0 of a smooth k-variety X and $\Omega^*(X)$. The technical construction of pullbacks is the subject of two talks. At the end one presents the state of advances on the conjectural isomorphism between Levine-Morel construction of algebraic cobordism and the "homotopical algebraic cobordism", the cohomology theory represented by motivic Thom spectrum in the Morel-Voevodsky $\mathbb{A}^1$-stable homotopy category.
The *Arbeitsgemeinschaft was organised by Marc Levine(Boston) and Fabien Morel(München). It was well attended with over 40 participants.*

Mathematics Subject Classification (2000): 19E15.

Introduction by the Organisers

Over the years, many different types and flavors of cohomology theories for algebraic varieties have been constructed. Theories like étale cohomology or de Rham cohomology provide algebraic versions of the topological theory of singular cohomology. The Chow ring and algebraic K_0 are other (partial) examples, more directly tied to algebraic geometry.

The partial theory K_0^{alg} was extended to a full theory with the advent of Quillen's higher algebraic K-theory. It took considerably longer for the Chow ring to be extended to motivic cohomology. In the process of doing so, Voevodsky developed his category of motives, and this construction was put in a more general setting with the development by Morel-Voevodskyof of $\mathbb{A}^1$ homotopy theory. This enabled a systematic construction of cohomology theories on algebraic varieties, with algebraic K-theory and motivic cohomology being only two fundamental examples. These two cohomology theories have in common the existence of a good theory of push-forward maps for projective morphisms. Not all cohomology theories have this structure, those that do are called *oriented*. In the Morel-Voevodsky stable homotopy category, the universal oriented theory is represented by the $\mathbb{P}^1$-spectrum $M\mathbb{G}\ell$, an algebraic version of the classical Thom spectrum MU. The corresponding cohomology theory $M\mathbb{G}\ell^{*,*}$ is called *higher algebraic cobordism*.

In an attempt to better understand the theory $M\mathbb{G}\ell^{*,*}$, Levine and Morel constructed a theory of *algebraic cobordism* Ω^*. This is (conjecturally) related to $M\mathbb{G}\ell^{*,*}$ as the classical Chow ring CH^* is to motivic cohomology and like CH^*, Ω^* has a purely algebro-geometric description. In addition to giving some insight into $M\mathbb{G}\ell^{*,*}$, Ω^* gives a simultaneous presentation of both CH^* and K_0, exhibiting K_0 as a deformation of CH^*. Ω^* has also been used to give conceptually simple proofs of various "degree formulas" first formulated by Rost. These degree formulas have been used in the study of Pfister quadrics and norm varieties, properties of which are used in the proofs of the Milnor conjecture and the Bloch-Kato conjecture.

In this workshop, we describe aspects of the topological theory of complex cobordism which are important for algebraic cobordism (Lectures 1-3) and give the construction of Ω^* and proofs of its fundamental properties (Lectures 4-7). In lectures 8-11, we show how K_0 and CH^* are described by Ω^*, how Ω^* recovers the universal formal group law, give the proof the generalized degree formula for Ω^* and use this to proof the degree formula for the Segre class. Additional applications to Steenrod operations, further degree formulas and the use of these in the study of quadrics and other varietes is given in lectures 12 and 13. Lectures 14 and 15 concern the construction of funtorial pull-backs in algebraic cobordism. The two concluding lectures (16 and 17) give a quick sketch of the Morel-Voevodsky $\mathbb{A}^1$ stable homotopy category and describe what we know about $M\mathbb{G}\ell$ and its relation to motivic cohomology and Ω^*.

The workshop *Algebraic Cobordism*,organised by Marc Levin (Boston) and Fabien Morel (München) was held April 4th–April 8th, 2005. This meeting was well attended with 55 participants.