Preface

I think that the reason for the usefulness of mathematics in reality – not just physical, but also biological, economic, etc. – is a mystery.

Ennio De Giorgi

We meet optimization problems daily: to maximize wealth, to maximize an income, to minimize a loss, to minimize time, to minimize distances, etc. Optimization, in turn, is based on mathematics which, for many students, presents a hurdle in their carrier. The underlying theory involves highly sophisticated tools for applications, and a beginner may be lost in too complicated formalisms. This fact is even more evident in time-dependent processes, where the variables we have at our disposal to maximize or minimize a given cost affect the whole evolution of a system; then, understanding even the form that Fermat's rule or Lagrange[1] multipliers should take becomes a conundrum. The purpose of this book is to gently help the reader get acquainted with the basic notions and methods that are of common use in this branch of optimization theory, called *dynamic optimization*.

The book aims to be readable at two different velocities, in order to be exploited both by mathematics students and by students which merely need dynamic optimization for applied sciences, economy and data science. For this reason, the book is self-contained, needing no prerequisites, and describes applications as well as a large number of examples and exercises. Simple exercises are given throughout the text, with or without hints, and more involved exercises with full solutions can be found in the final chapter. It is a "friendly book" for students, who nowadays are often in a hurry and, when they do not remember some arguments, they look on the web. We are confident that the readers of our book will not need to look on the web, they simply have to look at the right place within the book.

Chapter 1 contains a survey of ordinary differential equations at a sophomore level with some attention on specific equations that are needed in the other chapters. It can be considered as being totally made of prerequisites, inserted in the book for students that "forgot" what they learned about differential equations. Chapter 1 can also be regarded as a hard copy handbook, with no need to seek former notes or help from the web.

Functional analysis, the most theoretical part, is introduced lightly in Chapter 2, avoiding excessive formalism and full proofs. Some parts, such as the Fourier series, are again at sophomore level but the remaining parts introduce all the necessary tools (and not more than needed) for the optimization part.

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[1]Throughout this book we refer to Lagrange but, as a matter of fact, he was born in Torino (Italy) and his real name was Giuseppe Luigi Lagrangia (1736–1813).

Chapter 3 is devoted to the introduction of first-order partial differential equations, mostly focused on the method of characteristics and to the specific equations that appear in dynamic programming. This topic is usually covered in advanced maths courses but also complemented with a full theoretical background. The purpose here is to introduce only the part needed for dynamic programming.

Chapter 4 contains the fundamentals of the calculus of variations for functions of one variable, which is one of the objectives of this book. This material is usually contained in advanced courses, possibly within the framework of Sobolev spaces. In this book, we manage discussing this topic in the classical framework of spaces of continuous (and continuously differentiable) functions.

In the next two chapters, we extend the analysis to the second objective of this book, namely control theory. We start in Chapter 5, deriving the main controllability properties of linear systems and illustrating them with examples. Notice that, in this chapter, we have chosen to focus on controllability rather than optimal control, even though it is often the case that the control we construct to steer an initial state to a final one turns out to be the one that does so with minimal use of energy. Then, in Chapter 6, we quickly move into optimal control for general nonlinear systems focussing on necessary conditions for optimality, the celebrated Pontryagin Minimum (or Maximum) Principle, which still represents the main tool for detecting optimal controls. In order not to overwhelm the reader with technical details, we proceed from simpler settings to more general ones developing the various forms that PMP can take in a progressive way. We study problems with fixed horizon and no terminal conditions first. Then, we let the horizon vary as one of the control parameters and we impose a terminal condition on the states. In the latter case, we restrict the exposition to control systems with affine dependence on controls. Two main examples are studied in full detail: the classical moon lander problem and a new lockdown strategy for pandemic viruses.

Chapter 7 completes our treatment of dynamic optimization with an essential introduction to dynamic programming. This is, in our opinion, one of the most fascinating features of optimal control, as a strong connection is established between two distinct domains such as optimization and partial differential equations. We have kept the analysis to a very basic level, favouring those results that have a direct applications to optimal control problems. For this reason, we have chosen not to enter the subject of viscosity solutions, that would build the theory of Hamilton–Jacobi equations on much more general grounds but would require further elements of nonsmooth analysis to address applications. On the other hand, from this book the reader will collect all necessary prerequisites to pursue further reading, such as [8], on the connections of Hamilton–Jacobi equations with control problems.

Finally, Chapter 8 contains a large number of solved problems related to all the subjects of this book.

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