Preface

Many partial differential equations (PDEs) arising in physics can be seen as infinite-dimensional Hamiltonian systems

$$\partial_t z = J(\nabla_z H)(z), \quad z \in E, \tag{0.0.1}$$

where the Hamiltonian function $H\colon E \to \mathbb{R}$ is defined on an *infinite*-dimensional Hilbert space E of functions $z := z(x)$, and J is a nondegenerate antisymmetric operator.

Some main examples are the *nonlinear wave* (NLW) equation

$$u_{tt} - \Delta u + V(x)u + g(x,u) = 0, \tag{0.0.2}$$

the nonlinear Schrödinger (NLS) equation, the beam equation, the higher-dimensional membrane equation, the water-waves equations, i.e., the Euler equations of hydrodynamics describing the evolution of an incompressible irrotational fluid under the action of gravity and surface tension, as well as its approximate models like the Korteweg de Vries (KdV), Boussinesq, Benjamin–Ono, and Kadomtsev–Petviashvili (KP) equations, among many others. We refer to [102] for a general introduction to Hamiltonian PDEs.

In this monograph we shall adopt a "dynamical systems" point of view regarding the NLW equation (0.0.2) equipped with periodic boundary conditions $x \in \mathbb{T}^d := (\mathbb{R}/2\pi\mathbb{Z})^d$ as an infinite-dimensional Hamiltonian system, and we shall prove the existence of Cantor families of finite-dimensional invariant tori filled by quasiperiodic solutions of (0.0.2). The first results in this direction were obtained by Bourgain [42]. The search for invariant sets for the flow is an essential change of paradigm in the study of hyperbolic equations with respect to the more traditional pursuit of the initial value problem. This perspective has allowed the discovery of many new results, inspired by finite-dimensional Hamiltonian systems, for Hamiltonian PDEs.

When the space variable x belongs to a *compact* manifold, say $x \in [0,\pi]$ with Dirichlet boundary conditions or $x \in \mathbb{T}^d$ (periodic boundary conditions), the dynamics of a Hamiltonian PDE (0.0.1), like (0.0.2), is expected to have a "recurrent" behavior in time, with many *periodic* and *quasiperiodic* solutions, i.e., solutions (defined for *all* times) of the form

$$u(t) = U(\omega t) \in E \quad \text{where} \quad \mathbb{T}^\nu \ni \varphi \mapsto U(\varphi) \in E \tag{0.0.3}$$

is continuous, 2π-periodic in the angular variables $\varphi := (\varphi_1, \dots, \varphi_\nu)$ and the frequency vector $\omega \in \mathbb{R}^\nu$ is nonresonant, namely $\omega \cdot \ell \neq 0$, $\forall \ell \in \mathbb{Z}^\nu \setminus \{0\}$. When $\nu = 1$, the solution $u(t)$ is periodic in time, with period $2\pi/\omega$. If $U(\omega t)$ is a quasiperiodic solution then, since the orbit $\{\omega t\}_{t\in\mathbb{R}}$ is *dense* on $\mathbb{T}^\nu$, the torus manifold $U(\mathbb{T}^\nu) \subset E$ is invariant under the flow of (0.0.1).

Note that the linear wave equation (0.0.2) with $g = 0$,

$$u_{tt} - \Delta u + V(x)u = 0, \quad x \in \mathbb{T}^d , \tag{0.0.4}$$

possesses many quasiperiodic solutions. Indeed the self-adjoint operator $-\Delta + V(x)$ has a complete L^2 orthonormal basis of eigenfunctions $\Psi_j(x)$, $j \in \mathbb{N}$, with eigenvalues $\lambda_j \to +\infty$,

$$(-\Delta + V(x))\,\Psi_j(x) = \lambda_j\,\Psi_j(x), \quad j \in \mathbb{N}\,. \tag{0.0.5}$$

Suppose for simplicity that $-\Delta + V(x) > 0$, so that the eigenvalues $\lambda_j = \mu_j^2$, $\mu_j > 0$, are positive, and *all* the solutions of (0.0.4) are

$$\sum_{j \in \mathbb{N}} \alpha_j \cos(\mu_j t + \theta_j)\Psi_j(x), \quad \alpha_j,\ \theta_j \in \mathbb{R}\,, \tag{0.0.6}$$

which, depending on the resonance properties of the linear frequencies $\mu_j = \mu_j(V)$, are periodic, quasiperiodic, or almost-periodic in time (i.e., quasiperiodic with infinitely many frequencies).

What happens to these solutions under the effect of the nonlinearity $g(x,u)$?

There exist special nonlinear equations for which all the solutions are still periodic, quasiperiodic, or almost-periodic in time, for example the KdV, Benjamin–Ono, and 1-dimensional defocusing cubic NLS equations. These are completely integrable PDEs. However, for generic nonlinearities, one expects, in analogy with the celebrated Poincaré nonexistence theorem of prime integrals for nearly integrable Hamiltonian systems, that this is not the case.

On the other hand, for sufficiently small Hamiltonian perturbations of a nondegenerate integrable system in $\mathbb{T}^n \times \mathbb{R}^n$, the classical Kolmogorov–Arnold–Moser (KAM) theorem proves the persistence of quasiperiodic solutions with *Diophantine* frequency vectors $\omega \in \mathbb{R}^n$, i.e., vectors satisfying for some $\gamma > 0$ and $\tau \geq n-1$, the nonresonance condition

$$|\omega \cdot \ell| \geq \frac{\gamma}{|\ell|^\tau}, \quad \forall \ell \in \mathbb{Z}^n \setminus \{0\}\,. \tag{0.0.7}$$

Such frequencies form a Cantor set of $\mathbb{R}^n$ of positive measure if $\tau > n-1$. These quasiperiodic solutions (which densely fill invariant Lagrangian tori) were constructed by Kolmogorov [99] and Arnold [2] for analytic systems using an iterative Newton scheme, and by Moser [105–107] for differentiable perturbations by introducing smoothing operators. This scheme then gave rise to abstract Nash–Moser implicit function theorems like the ones proved by Zehnder [128, 129] (see also [109] and Section 2.1).

What happens for infinite-dimensional systems like PDEs?

- The central question of KAM theory for PDEs is: *do "most" of the periodic, quasiperiodic, or almost-periodic solutions of an integrable PDE (linear or nonlinear) persist, just slightly deformed, under the effect of a nonlinear perturbation?*

KAM theory for PDEs started a bit more than thirty years ago with the pioneering works of Kuksin [100] and Wayne [126] on the existence of quasiperiodic solutions for semilinear perturbations of 1-dimensional linear wave and Schrödinger equations in the interval $[0, \pi]$. These results are based on an extension of the KAM perturbative approach developed for the search for lower-dimensional tori in finite-dimensional systems (see [56, 107, 111]) and relies on the verification of the so-called second-order Melnikov nonresonance conditions.

Nowadays KAM theory for 1-dimensional PDEs has reached a satisfactory level of comprehension concerning quasiperiodic solutions, while questions concerning almost-periodic solutions remain quite open. The known results include bifurcation of small-amplitude solutions [17,103,113], perturbations of large finite gap solutions [29,34,97,101,102], extension to periodic boundary conditions [36,48,54,71], use of weak nondegeneracy conditions [12], nonlinearities with derivatives [19,93,130] up to quasilinear ones [6–8,65], including water-waves equations [5,32], applications to quantum harmonic oscillators [10,11,82], and a few examples of almost-periodic solutions [43,114]. We describe these developments in more detail in Chapter 2.

Also, KAM theory for multidimensional PDEs still contains few results and a satisfactory picture is under construction. If the space dimension d is two or more, major difficulties are:

1. The eigenvalues μ_j^2 of the Schrödinger operator $-\Delta + V(x)$ in (0.0.5) appear in huge clusters of increasing size. For example, if $V(x) = 0$, and $x \in \mathbb{T}^d$, they are
$$|k|^2 = k_1^2 + \cdots + k_d^2, \quad k = (k_1, \ldots, k_d) \in \mathbb{Z}^d.$$
2. The eigenfunctions $\Psi_j(x)$ may be "not localized" with respect to the exponentials. Roughly speaking, this means that there is no one-to-one correspondence $h : \mathbb{N} \to \mathbb{Z}^d$ such that the entries $(\Psi_j, e^{\mathrm{i}k\cdot x})_{L^2}$ of the change-of-basis matrix (between $(\Psi_j)_{j\in\mathbb{N}}$ and $(e^{\mathrm{i}k\cdot x})_{k\in\mathbb{Z}^d}$) decay rapidly to zero as $|k - h(j)| \to \infty$.

The first existence result of time-periodic solutions for the NLW equation
$$u_{tt} - \Delta u + mu = u^3 + \text{h.o.t.}, \quad x \in \mathbb{T}^d,\ d \geq 2,$$
was proved by Bourgain in [37], by extending the Craig–Wayne approach [54], originally developed if $x \in \mathbb{T}$. Further existence of results of periodic solutions have been proved by Berti and Bolle [22] for merely differentiable nonlinearities, Berti, Bolle, and Procesi [26] for Zoll manifolds, Gentile and Procesi [76] using Lindstedt series techniques, and Delort [55] for NLS equations using paradifferential calculus.

The first breakthrough result about the existence of quasiperiodic solutions for space multidimensional PDEs was proved by Bourgain [39] for analytic Hamiltonian NLS equations of the form
$$\mathrm{i}u_t = \Delta u + M_\sigma u + \varepsilon \partial_{\bar{u}} H(u, \bar{u}) \tag{0.0.8}$$
with $x \in \mathbb{T}^2$, where $M_\sigma = \mathrm{Op}(\sigma_k)$ is a Fourier multiplier supported on finitely many sites $\mathbb{E} \subset \mathbb{Z}^2$, i.e., $\sigma_k = 0$, $\forall k \in \mathbb{Z}^2 \setminus \mathbb{E}$. The σ_k, $k \in \mathbb{E}$, play the role of

external parameters used to verify suitable nonresonance conditions. Note that the eigenfunctions of $\Delta + M_\sigma$ are the exponentials $e^{\mathrm{i}k\cdot x}$ and so the above-mentioned problem 2 is not present.

Later, using subharmonic analysis tools previously developed for quasiperiodic Anderson localization theory by Bourgain, Goldstein, and Schlag [44], Bourgain was able in [41, 42] to extend this result in any space dimension d, and also for NLW equations of the form

$$u_{tt} - \Delta u + M_\sigma u + \varepsilon F'(u) = 0, \quad x \in \mathbb{T}^d\,, \tag{0.0.9}$$

where $F(u)$ is a polynomial in u. Here $F'(u)$ denotes the derivative of F. We also mention the existence results of quasiperiodic solutions by Bourgain and Wang [45, 46] for NLS and NLW equations under a random perturbation. The stochastic case is a priori easier than the deterministic one because it is simpler to verify the nonresonance conditions with a random variable.

Quasiperiodic solutions $u(t,x) = U(\omega t, x)$ of (0.0.9) with a frequency vector $\omega \in \mathbb{R}^\nu$, namely solutions $U(\varphi, x)$, $\varphi \in \mathbb{T}^\nu$, of

$$(\omega \cdot \partial_\varphi)^2 U - \Delta U + M_\sigma U + \varepsilon F'(U) = 0\,, \tag{0.0.10}$$

are constructed by a Newton scheme. The main analysis concerns finite-dimensional restrictions of the quasiperiodic operators obtained by linearizing (0.0.10) at each step of the Newton iteration,

$$\Pi_N \big((\omega \cdot \partial_\varphi)^2 - \Delta + M_\sigma + \varepsilon b(\varphi, x)\big)_{|\mathcal{H}_N}\,, \tag{0.0.11}$$

where $b(\varphi, x) = F''(U(\varphi, x))$ and Π_N denotes the projection on the finite-dimensional subspace

$$\mathcal{H}_N := \Big\{ h = \sum_{|(\ell,k)|\le N} h_{\ell,k} e^{\mathrm{i}(\ell\cdot\varphi + k\cdot x)}, \ell \in \mathbb{Z}^\nu, k \in \mathbb{Z}^d \Big\}\,.$$

The matrix that represents (0.0.11) in the exponential basis is a perturbation of the diagonal matrix $\mathrm{Diag}(-(\omega\cdot\ell)^2 + |k|^2 + \sigma_k)$ with off-diagonal entries $\varepsilon(\widehat{b}_{\ell-\ell',k-k'})$ that decay exponentially to zero as $|(\ell-\ell', k-k')| \to +\infty$, assuming that b is analytic (or subexponentially, if b is Gevrey). The goal is to prove that such a matrix is invertible for most values of the external parameters, and that its inverse has an exponential (or Gevrey) off-diagonal decay. It is not difficult to impose lower bounds for the eigenvalues of the self-adjoint operator (0.0.11) for most values of the parameters. These first-order Melnikov nonresonance conditions are essentially the minimal assumptions for proving the persistence of quasiperiodic solutions of (0.0.9), and provide estimates of the inverse of the operator (0.0.11) in L^2-norm. In order to prove fast off-diagonal decay estimates for the inverse matrix, Bourgain's technique is a "multiscale" inductive analysis based on the repeated use of the "resolvent identity." An essential ingredient is that the "singular" sites

$$(\ell, k) \in \mathbb{Z}^\nu \times \mathbb{Z}^d \quad \text{such that} \quad \left|-(\omega \cdot \ell)^2 + |k|^2 + \sigma_k\right| \leq 1 \tag{0.0.12}$$

are separated into clusters that are sufficiently distant from one another. However, the information (0.0.12) about just the linear frequencies of (0.0.9) is not sufficient (unlike for time-periodic solutions [37]) in order to prove that the inverse matrix has an exponential (or Gevrey) off-diagonal decay. Also, finer nonresonance conditions at each scale along the induction need to be verified. We describe the multiscale approach in Section 2.4 and we prove novel multiscale results in Chapter 5.

These techniques have been extended in the recent work of Wang [125] for the nonlinear Klein–Gordon equation

$$u_{tt} - \Delta u + u + u^{p+1} = 0, \quad p \in \mathbb{N}, \ x \in \mathbb{T}^d,$$

that, unlike (0.0.9), is parameter independent. A key step is to verify that suitable nonresonance conditions are fulfilled for most "initial data." We refer to [124] for a corresponding result for the NLS equation.

Another stream of important results for multidimensional PDEs were inaugurated in the breakthrough paper by Eliasson and Kuksin [61] for the NLS equation (0.0.8). In this paper the authors are able to block diagonalize, and reduce to constant coefficients, the quasiperiodic Hamiltonian operator obtained at each step of the iteration. This KAM reducibility approach extends the perturbative theory developed for 1-dimensional PDEs, by verifying the so-called second-order Melnikov nonresonance conditions. It allows one to also prove directly the linear stability of the quasiperiodic solutions. Other results in this direction have been proved for the 2-dimensional-cubic NLS equation by Geng, Xu, and You [75], in any space dimension and arbitrary polynomial nonlinearities by Procesi and Procesi [117, 118], and for beam equations by Geng and You [72] and Eliasson, Grébert, and Kuksin [58]. Unfortunately, the second-order Melnikov conditions are violated for NLW equations for which an analogous reducibility result does not hold. We describe the KAM reducibility approach with PDE applications in Sections 2.2 and 2.3.

We now present more in detail the goal of this research monograph. The main result is the existence of small-amplitude time-quasiperiodic solutions for autonomous NLW equations of the form

$$\begin{gathered} u_{tt} - \Delta u + V(x)u + g(x,u) = 0\,, \\ x \in \mathbb{T}^d, \quad g(x,u) = a(x)u^3 + O(u^4)\,, \end{gathered} \tag{0.0.13}$$

in *any* space dimension $d \geq 1$, where $V(x)$ is a smooth *multiplicative* potential such that $-\Delta + V(x) > 0$, and the nonlinearity is C^∞. Given a finite set $\mathbb{S} \subset \mathbb{N}$ (tangential sites), we construct quasiperiodic solutions $u(\omega t, x)$ with frequency vector $(\omega_j)_{j \in \mathbb{S}}$, of the form

$$u(\omega t, x) = \sum_{j \in \mathbb{S}} \alpha_j \cos(\omega_j t)\Psi_j(x) + r(\omega t, x), \quad \omega_j = \mu_j + O(|\alpha|)\,, \tag{0.0.14}$$

where the remainder $r(\varphi, x)$ is $o(|\alpha|)$-small in some Sobolev space; here $\alpha := (\alpha_j)_{j \in \mathbb{S}}$. The solutions (0.0.14) are thus a small deformation of linear solutions (0.0.6), supported on the "tangential" space spanned by the eigenfunctions $(\Psi_j(x))_{j \in \mathbb{S}}$, with a much smaller component in the normal subspace. These quasiperiodic solutions of (0.0.13) exist for generic potentials $V(x)$, coefficients $a(x)$, and "most" small values of the amplitudes $(\alpha_j)_{j \in \mathbb{S}}$. The precise statement is given in Theorems 1.2.1 and 1.2.3.

The proof of this result requires various mathematical methods that this book aims to present in a systematic and self-contained way. A complete outline of the steps of the proof is presented in Section 2.5. Here we just mention that we shall use a Nash–Moser iterative scheme in scales of Sobolev spaces for the search for an invariant torus embedding supporting quasiperiodic solutions, with a frequency vector ω to be determined. One key step is to establish the existence of an approximate inverse for the operators obtained by linearizing the NLW equation at any approximate quasiperiodic solution $u(\omega t, x)$, and to prove that such an approximate inverse satisfies tame estimates in Sobolev spaces, with loss of derivatives due to the small divisors. These linearized operators have the form

$$h \mapsto (\omega \cdot \partial_\varphi)^2 h - \Delta h + V(x) h + (\partial_u g)(x, u(\omega t, x)) h$$

with coefficients depending on $x \in \mathbb{T}^d$ and $\varphi \in \mathbb{T}^{|\mathbb{S}|}$. The construction of an approximate inverse requires several steps. After writing the wave equation as a Hamiltonian system in infinite dimensions, the first step is to use a symplectic change of variables to approximately decouple the tangential and normal components of the linearized operator. This is a rather general procedure for autonomous Hamiltonian PDEs, which reduces the problem to the search for an approximate inverse for a quasiperiodic Hamiltonian linear operator acting in the subspace normal to the torus (see Chapter 7 and Appendix C).

In order to avoid the difficulty posed by the violation of the second-order Melnikov nonresonance conditions required by a KAM reducibility scheme, we develop a multiscale inductive approach à la Bourgain, which is particularly delicate since the eigenfunctions $\Psi_j(x)$ of $-\Delta + V(x)$ defined in (0.0.5) are not localized near the exponentials. In particular, the matrix elements $(\Psi_j, a(x)\Psi_{j'})_{L^2}$ representing the multiplication operator with respect to the basis of the eigenfunctions $\Psi_j(x)$ do not decay, in general, as $j - j' \to \infty$. In Chapter 5 we provide the complete proof of the multiscale proposition (which is fully self-contained together with Appendix B), which we shall use in Chapters 10 and 11. These results extend the multiscale analysis developed for forced NLW and NLS equations in [23] and [24].

The presence of a multiplicative potential $V(x)$ in (0.0.13) also makes it difficult to control the variations of the tangential and normal frequencies due to the effect of the nonlinearity $a(x)u^3 + O(u^4)$ with respect to parameters. In this monograph, after a careful bifurcation analysis of the quasiperiodic solutions, we are able to use just the length $|\omega|$ of the frequency vector as an internal parameter to verify all the nonresonance conditions along the iteration. The frequency vector is constrained to a fixed direction, see (1.2.24) and (1.2.25). The measure estimates rely on positivity arguments for the variation of parameter dependent families of self-adjoint matrices, see Section 5.8. These properties (see (9.1.8)) are verified for the linearized operators obtained along the iteration.

The genericity of the nonresonance and nondegeneracy conditions that we require on the potential $V(x)$ and the coefficient $a(x)$ in the nonlinearity $a(x)u^3 + O(u^4)$, are finally verified in Chapter 13.

The techniques developed above for the NLW equation (0.0.13) could certainly be used to prove a corresponding result for NLS equations. However we have decided to focus on the NLW equation because, as explained above, there are fewer results available in this case. This context seems to better illustrate the advantages of the present approach in comparison to that of reducibility.

A feature of the monograph is to present the proofs, techniques, and ideas developed in a self-contained and expanded manner, with the hope to enhance further developments. We also aim to describe the connections of this result with previous works in the literature. The techniques developed in this monograph have deep connections with those used in Anderson-localization theory and we hope that the detailed presentation in this manuscript of all technical aspects of the proofs will allow a deeper interchange between the Anderson-localization and KAM for PDEs scientific communities.

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