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Classification of complex algebraic surfaces. (English)

EMS Series of Lectures in Mathematics. Berlin: European Mathematical Society (EMS) (ISBN 978-3-03719-210-8/pbk; 978-3-03719-710-3/ebook). ix, 133 p. (2020).

In this greatly written monograph the author, one of few world-class experts on algebraic surfaces, provides a detailed introduction to the classification of algebraic surfaces from a viewpoint of Mori's programme. One of the most important subjects in the theory of algebraic varieties is the classification problem. In this context, the first fundamental question is about the rationality. Let us recall that for complex algebraic curves their rationality is described by the vanishing of the genus. In the case of algebraic surfaces, the Enriques-Kodaira classification, which origins date back to 1914, is the classical approach which provides a classification of smooth complex compact surfaces into ten classes. In order to obtain this classification, one needs to cleverly use the notion of the Kodaira dimension and the Hodge numbers of algebraic surfaces. Let X be a smooth complex compact surface and denote by $g_i(X)$ the dimension of the space of holomorphic i-forms on X. Denote by K_X the canonical line bundle of X. Then the n-th plurigenera $P_n(X)$ of X is defined as

$$P_n(X) = \dim, H^0(X, K_X^{\otimes n}) \text{ for } n \geq 1.$$

In particular, $P_2(X) = g_2(X)$. One of the most crucial things is that $P_n(X)$'s are birational invariants of X. It is an immediate observation that for a rational surface X we have that all $P_n(X) = 0$ vanish. The first step in the classification programme of algebraic surfaces is the following result due to Castelnuovo.
Theorem. (Castelnuovo's criterion). An algebraic surface X is rational if and only if $g_1(X) = P_2(X) = 0$.
Now if we define a ruled surface as a surface which is birationally equivalent to the product of $\mathbb{P}^1$ and a curve C, then we have an interesting characterization of these surfaces.
Theorem. (Castelnuovo-Enriques). An algebraic surface X is ruled if and only if $P_{12}(X) = 0$.
The idea of classifying algebraic surfaces, according to Enriques, boils down to studying the behaviour of $P_n(X)$ provided that $n \to \infty$. According to this approach, we have four types of algebraic surfaces. In the first class we have those surfaces X for which all $P_n(X)$'s are equal to zero (these are rational surfaces). In the second class we have surfaces for which $P_n(X)$ is equal to either 0 or 1, but at least one $P_m(X) \neq 0$ for a certain m. In the third class we have surfaces that can be characterized by $P_n(X) \sim O(n)$, and in the last class we have surfaces for which $P_n(X) \sim O(n^2)$.
In the monograph under review, the author provides a modern treatment on the classification problem of algebraic surfaces using Mori's ideas, well-known as Mori's theory of minimal models. One of the key results (for algebraic surfaces over the complex numbers) which allows to construct minimal models is Castelnuovo's contraction theorem.
Theorem. (Castelnuovo's contraction criterion). Let X be a smooth complex projective surface and $C \subset X$ a smooth irreducible curve. Then there is a morphism $\pi : X \to Y$ with Y a smooth surface and a point $p \in Y$ such that X is the blow-up of Y at p with the exceptional divisor C if and only if $C \simeq \mathbb{P}^1$ and $C^2 = -1$.
It is a good point to emphasize that the author does not restrict his attention only to surfaces defined over the complex numbers. There are results which hold in positive characteristic – in this context it is worth mentioning that the $P_{12}(X)$ theorem provided in Chapter 12 is characteristic free.

Let us briefly present an outline of the content of the monograph. In the first chapter, the author provides a crashcourse on tools and techniques that one need to use in the classification problem(s). In the second chapter, the author recalls a classical characterization of the complex projective plane. The third chapter is crucial and the author introduce there the notion of minimal models emphasizing the role of blow-downs and Castelnuovo's contraction theorem. In the fourth chapter, a complete classification of ruled surfaces is provided. In Chapter 5, the author provides an outline of the tools needed to understand surfaces with non-nef canonical bundles, especially the rationality theorem. The whole Chapter 6 is devoted to the Cone Theorem, where we can also find a detailed proof. In Chapter 7, the minimal model programme is introduced, and the main result of this chapter provides the uniqueness of the strong minimal models. In Chapter 8, the author proves the Castelnuovo rationality criterion. In Chapter 9, the fundamental classification result for algebraic surfaces with K_X nef due to Castelnuovo, de Franchis, and Enriques is explained. Chapter 10 is devoted to the abundance theorem and its consequences for algebraic surfaces. In Chapter 11, the author discusses surfaces of general type. The last chapters are non-standard (if we compare this part of the monograph with standard/classical textbooks on surfaces). Firstly, the author presents the Bagnera-De Franchis classification of bielliptic (or hyperelliptic surfaces). In Chapter 13 the author discusses the $P_{12}(X)$-theorem in the setting provided by Catanese and Li work, namely Enriques' classification of algebraic surface in characteristic $p > 0$ – this is a very remarkable result. In Chapter 14 the author provides an outline of the Sarkisov programme for algebraic surfaces. In Chapter 15, the classical Noether-Castelnuovo theorem on the group of birational transformations of $\mathbb{P}^2$ is discussed. Finally, the last chapter of the monograph is devoted to negative curves on algebraic surfaces, here the special role is played by the blow-ups of the complex projective plane at general points.

The monograph provides a quick and detailed introduction to the theory of algebraic surfaces. The modern treatment provided by the author is very useful for readers and it delivers the first (complete) step for Mori's programme of minimal models of algebraic varieties. I believe that the monograph is not designed for readers that do not possess a good knowledge of basics on algebraic geometry. However, this is not a serious prerequisite since every modern textbook devoted to the general introduction to algebraic geometry suffices to clear the knowledge gap. I can honestly recommend this monograph to everyone who wants to study and learn about the classification problems for algebraic surfaces in detail.

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