Introduction

In order to study a given group G, it is natural to look for mathematical objects on which G acts by automorphisms. For instance, in ordinary representation theory, one considers vector spaces on which G acts linearly. In topology, one might prefer topological spaces and continuous G-actions. In functional analysis, it might be operator algebras on which G is expected to act. And so on, and so forth. Those 'G-equivariant objects' usually assemble into a category, that we shall denote $\mathcal{M}(G)$. Constructing such categories $\mathcal{M}(G)$ of G-equivariant objects in order to study the group G is a simple but powerful idea. It is used in all corners of what we shall loosely call 'equivariant mathematics'.

In this work, we focus on *finite groups* G and *additive categories* $\mathcal{M}(G)$, *i.e.* categories in which one can add objects and add morphisms. Although topological or analytical examples may not seem very additive at first sight, they can be included in our discussion by passing to stable categories. Thus, to name a few explicit examples of such categories $\mathcal{M}(G)$, let us mention categories of $\Bbbk G$-modules $\mathcal{M}(G) = \mathrm{Mod}(\Bbbk G)$ or their derived categories $\mathcal{M}(G) = \mathrm{D}(\Bbbk G)$ in classical representation theory over a field $\Bbbk$, homotopy categories of G-spectra $\mathcal{M}(G) = \mathrm{SH}(G)$ in equivariant homotopy theory, and Kasparov categories $\mathcal{M}(G) = \mathrm{KK}(G)$ of G-C^*-algebras in noncommutative geometry. As the reader surely realizes at this point, the list of such examples is virtually endless: just let G act wherever it can! In fact, the entire Chapter 4 of this book is devoted to a review of examples.

Let us try to isolate the properties that such categories $\mathcal{M}(G)$ have in common. First of all, it is clear that in all situations we can easily construct a similar category $\mathcal{M}(H)$ for any other group H, in particular for subgroups $H \le G$. The variance of $\mathcal{M}(G)$ in the group G, through restriction, induction, conjugation, etc, is the bread and butter of equivariant mathematics. It is then a natural question to axiomatize what it means to have a reasonable collection of additive categories $\mathcal{M}(G)$ indexed by finite groups G, with all these links between them. In view of the ubiquity of such structures, it is somewhat surprising that such an axiomatic treatment did not appear earlier.

In fact, a lot of attention has been devoted to a similar but simpler structure, involving abelian groups instead of additive categories. These are the so-called Mackey functors. Let us quickly remind the reader of this standard notion, going back to work of Green [Gre71] and Dress [Dre73] almost half a century ago.

An ordinary *Mackey functor* M involves the data of abelian groups $M(G)$ indexed by finite groups G. These $M(G)$ come with restriction homomorphisms $R^G_H\colon M(G) \to M(H)$, induction or transfer homomorphisms $I^G_H\colon M(H) \to M(G)$, and conjugation homomorphisms $c_x\colon M(H) \to M({}^xH)$, for $H \le G$ and $x \in G$. This data is subject to a certain number of rules, most of them rather intuitive. Among them, the critical rule is the *Mackey double-coset formula*, which says that for all $H, K \le G$

the following two homomorphisms $M(H) \to M(K)$ are equal:

$$R^G_K \circ I^G_H = \sum_{[x]\in K\backslash G/H} I^K_{K\cap {}^xH} \circ c_x \circ R^H_{K^x\cap H} \,. \tag{0.0.1}$$

These Mackey functors are quite useful in representation theory and equivariant homotopy theory. See Webb's survey [Web00] or Appendix B.

Let us return to our categories $\mathcal{M}(G)$ of 'objects with G-actions'. In most examples, these $\mathcal{M}(G)$ behave very much like ordinary Mackey functors, with the obvious difference that they involve additive categories $\mathcal{M}(G)$ instead of abelian groups $M(G)$, and additive functors between them instead of $\mathbb{Z}$-linear homomorphisms. Actually, truth be told, the homomorphisms appearing in ordinary Mackey functors are often mere shadows of additive functors with the same name (restriction, induction, etc) existing at the level of underlying categories.

In other words, to axiomatize our categories $\mathcal{M}(G)$ and their variance in G, we are going to *categorify* the notion of ordinary Mackey functor. Our first, very modest, contribution is to propose a name for these categorified Mackey functors $\mathcal{M}$. We call them

Mackey 2-functors.

We emphasize that we do not pretend to 'invent' Mackey 2-functors out of the blue. Examples of such structures have been around for a long time and are as ubiquitous as equivariant mathematics itself. So far, the only novelty is the snazzy name.

Our first serious task will consist in pinning down the precise definition of Mackey 2-functor. But without confronting the devil in the detail quite yet, the heuristic idea should hopefully be clear from the above discussion. In first approximation, a Mackey 2-functor $\mathcal{M}$ consists of the data of an additive category $\mathcal{M}(G)$ for each finite group G, together with further structure like restriction and induction functors, and subject to a Mackey formula at the categorical level. An important aspect of our definition is that we shall want $\mathcal{M}$ to satisfy

ambidexterity.

This means that induction is both left and right adjoint to restriction: For each subgroup $H \le G$, the restriction functor $\mathcal{M}(G) \to \mathcal{M}(H)$ admits a two-sided adjoint. In pedantic parlance, induction and 'co-induction' coincide in $\mathcal{M}$.

Once we start considering adjunctions, we inherently enter a 2-categorical world. We not only have categories $\mathcal{M}(G)$ and functors to take into account (0-layer and 1-layer) but we also have to handle natural transformations of functors (2-layer), at the very least for the units and counits of adjunctions. Similarly, our version of the Mackey formula will not involve an *equality* between homomorphisms as in (0.0.1) but an *isomorphism* between functors. This 2-categorical information is essential, and it distinguishes our Mackey 2-functors from a more naive notion of 'Mackey functor with values in the category of additive categories' (which would miss the adjunction between R^G_H and I^G_H for instance). This important 2-layer in the structure of a Mackey 2-functor also explains our choice of the name. Still, the reader who is not versed in the refinements of 2-category theory should not throw the towel in despair. Most of this book can be understood by keeping in mind the usual 2-category CAT of categories, functors and natural transformations.

* * *

In a nutshell, the purpose of this work is to

- lay the foundations of the theory of Mackey 2-functors
- justify this notion by a large catalogue of examples
- provide some first applications, and
- construct a ‘motivic’ approach.

Let us now say a few words of these four aspects, while simultaneously outlining the structure of the book. After the present gentle introduction, Chapter 1 will provide an expanded introduction with more technical details.

* * *

The first serious issue is to give a solid definition of Mackey 2-functor that simultaneously can be checked in examples and yet provides enough structure to prove theorems. This balancing act relies here on three components:

(1) A ‘light’ definition of Mackey 2-functor, to be found in Definition 1.1.7. It involves four axioms (Mack 1)–(Mack 4) that the data $G \mapsto \mathcal{M}(G)$ should satisfy. These four axioms are reasonably easy to verify in examples. Arguably the most important one, (Mack 4), states that $\mathcal{M}$ satisfies ambidexterity.

(2) A ‘heavier’ notion of *rectified* Mackey 2-functor, involving another six axioms (Mack 5)–(Mack 10). Taken together, those ten axioms make it possible to reliably prove theorems about (rectified) Mackey 2-functors. However, some of these six extra axioms can be unpleasant to verify in examples.

(3) A Rectification Theorem 1.2.1, which roughly says that there is always a way to modify the 2-layer of any Mackey 2-functor $G \mapsto \mathcal{M}(G)$ satisfying (Mack 1)–(Mack 4) so that the additional axioms (Mack 5)–(Mack 10) are satisfied as well. In particular, one does not have to verify (Mack 5)–(Mack 10) in examples.

An introduction to the precise definition of Mackey 2-functor is to be found in Section 1.1. The full treatment appears in Chapter 2. The motivation for the idea of rectification is given in Section 1.2, with details in Chapter 3.

A first application follows immediately from the Rectification Theorem, namely we prove that for any subgroup $H \leq G$, the category $\mathcal{M}(H)$ is a *separable extension* of $\mathcal{M}(G)$. This result provides a unification and a generalization of a string of results brought to light in [Bal15] and [BDS15], where we proved separability by an *ad hoc* argument in each special case. In the very short Section 2.4, we give a uniform proof that all (rectified) Mackey 2-functors $\mathcal{M}$ automatically satisfy this separability property. Conceptually, the problem is the following. How can we ‘carve out’ the category $\mathcal{M}(H)$ of H-equivariant objects over a subgroup from the category $\mathcal{M}(G)$ of G-equivariant objects over the larger group? The most naive guess would be to do the ‘carving out’ via localization. This basically never works, $\mathcal{M}(H)$ is almost never a localization of $\mathcal{M}(G)$, but separable extensions are the next best thing. Considering separable extensions instead of localizations is formally analogous to considering the étale topology instead of the Zariski topology in algebraic geometry. See further commentary on the meaning and relevance of separability in Section 1.3.

We return to the topic of applications below, when we comment on motives. For now, let us address the related question of examples. We discuss this point at some length because we consider the plethora of examples to be a great positive feature of the theory. Also, the motivic approach that we discuss next is truly justified by this very fact that Mackey 2-functors come in all shapes and forms.

It should already be intuitively clear from our opening paragraphs that Mackey 2-functors pullulate throughout equivariant mathematics. In any case, beyond this gut feeling that they should exist in many settings, a reliable source of rigorous examples of Mackey 2-functors can be found in the theory of *Grothendieck derivators* (see Groth [Gro13]). Our Ambidexterity Theorem 4.1.1 says that the restriction of an *additive* derivator to finite groups automatically satisfies the ambidexterity property making it a Mackey 2-functor. This result explains why it is so common in practice that induction and co-induction coincide in additive settings. It also provides a wealth of examples of Mackey 2-functors in different subjects. Let us emphasize this point: The theory of derivators itself covers a broad variety of backgrounds, in algebra, topology, geometry, etc. Furthermore, derivators can always be stabilized (see [Hel97] and [Col19]) and stable derivators are always additive. In other words, via derivators, that is, via general homotopy theory, we gain a massive collection of readily available examples of Mackey 2-functors from algebra, topology, geometry, etc. In particular, any stable Quillen model category $\mathcal{Q}$ provides a Mackey 2-functor $G \mapsto \mathcal{M}(G) := \mathrm{Ho}(\mathcal{Q}^G)$, via diagram categories. In the five decades since [Qui67], examples of Quillen model categories have been discovered in all corners of mathematics, see for instance Hovey [Hov99] or [HPS97]. An expanded introduction to these ideas can be found in Section 1.4.

But there is even more! In Sections 4.2 to 4.4, we provide further methods to handle trickier examples of Mackey 2-functors which cannot be obtained directly from a derivator. For instance, stable module categories (Proposition 4.2.5) in modular representation theory or genuine G-equivariant stable homotopy categories (Example 4.3.8) can be shown not to come from the restriction of a derivator to finite groups. Yet they are central examples of Mackey 2-functors and we explain how to prove this in Chapter 4.

* * *

Let us now say a word of the motivic approach, which is our most ambitious goal. It will occupy the lion's share of this work, namely Chapters 5 to 7. We now discuss these ideas for readers with limited previous exposure to motives. A more technical introduction can be found in Section 1.5.

In algebraic geometry, Grothendieck's *motives* encapsulate the common themes recurring throughout a broad range of 'Weil' cohomology theories. These cohomology theories are defined on algebraic varieties (*e.g.* on smooth projective varieties), take values in all sorts of different abelian categories, and are described axiomatically. Instead of algebraic varieties, we consider here finite groups. Instead of Weil cohomology theories, we consider of course Mackey 2-functors.

The motivic program seeks to construct an initial structure through which all other instances of the same sort of structure will factor. These ideas led Grothendieck to the plain 1-category of (pure) motives in algebraic geometry. Because of our added 2-categorical layer, the same philosophy naturally leads us to a

2-category of Mackey 2-motives.

The key feature of this 2-category is that every single Mackey 2-functor out there factors uniquely via Mackey 2-motives. The proof of this non-trivial fact is another application of the Rectification Theorem, together with some new constructions. Since we hammered the point that Mackey 2-functors are not mere figments of our imagination but very common structures, this factorization result applies broadly to many situations pre-dating our theory.

Perhaps this is a good place to further comment in non-specialized terms on the virtues of the motivic approach, beginning with algebraic geometry. The fundamental idea is of course the following. Since every Weil cohomology theory factors canonically via the category of motives, each result that can be established motivically will have a realization, an avatar, in every single example. Among the most successful such results are the so-called 'motivic decompositions'. In the motivic category, some varieties X decompose as a direct sum of other simpler motives. As a corollary, every single Weil cohomology theory evaluated at X will decompose into simpler pieces accordingly. The motivic decomposition happens entirely within the 'abstract' motivic world but the application happens wherever the Weil cohomology takes its values. And since Weil cohomology theories come in all shapes and forms, this type of result is truly powerful.

Let us see how this transposes to Mackey 2-motives. The overall pattern is the same. Whenever we find a motivic decomposition of the 2-motive of a given finite group G, we know in advance that *every single* Mackey 2-functor $\mathcal{M}(G)$ evaluated at that group will decompose into smaller pieces accordingly. Because of the additional 2-layer, things happen 'one level up', namely we decompose the identity 1-cell of G, which really amounts to decomposing the 2-motive G up to an equivalence (see the 'block decompositions' of A.7). Again, the range of applications is as broad as the list of examples of Mackey 2-functors.

In order to obtain concrete motivic decompositions, one needs to compute some endomorphism rings in the 2-category of Mackey 2-motives, more precisely the ring of 2-endomorphisms of the identity 1-cell Id_G of the Mackey 2-motive of G. Every decomposition of those rings, *i.e.* any splitting of the unit into sum of idempotents, will produce decompositions of the categories $\mathcal{M}(G)$ into 'blocks' corresponding to those idempotents.

In this direction, we prove in Chapter 7 that the above 2-endomorphism ring of Id_G is isomorphic to a ring already known to representation theorists, namely the so-called *crossed Burnside ring* of G introduced by Yoshida [Yos97]. See also Oda-Yoshida [OY01] or Bouc [Bou03]. The blasé reader should pause and appreciate the little miracle: A ring that we define through an a priori very abstract motivic construction turns out to be a ring with a relatively simple description, already known to representation theorists. It follows from this computation that every decomposition of the crossed Burnside ring yields a block decomposition of the Mackey 2-motive of G and therefore of *every* Mackey 2-functor evaluated at G, in every single example known today or to be discovered in the future.

* * *

This concludes the informal outline of this book. In addition to the seven main chapters mentioned above, we include two appendices. Appendix A collects all categorical prerequisites whereas Appendix B is dedicated to ordinary Mackey functors. We also draw the reader's attention to the extensive index at the very end, that will hopefully show useful in navigating the text.

A comparison with existing literature can be found in Section 1.6, after we introduce some relevant terminology in Chapter 1.

Acknowledgements: We thank Serge Bouc, Yonatan Harpaz, Ioannis Lagkas, Akhil Mathew, Hiroyuki Nakaoka, Beren Sanders, Stefan Schwede and Alexis Virelizier, for many motivating discussions and for technical assistance.

For the final version of this work, we are especially thankful to Serge Bouc, who recognized in an earlier draft a ring that was known to specialists as the crossed Burnside ring. Our revised Chapter 7 owes a lot to Serge's insight and to his generosity.

We are also grateful to an anonymous referee for their careful reading and helpful suggestions.

Finally, we are very grateful to Sylvia Fellmann and Thomas Hintermann at the EMS Publishing House, and to their typesetting team, for an efficient and friendly cooperation throughout the publication process.