

Chapter 1

Introduction

1.1 Introduction

Quantum Field Theory arose from the need to unify Quantum Mechanics with special relativity. It is usually formulated on the flat Minkowski spacetime, on which classical field equations, such as the Klein–Gordon, Dirac or Maxwell equations are easily defined. Their quantization rests on the so-called *Minkowski vacuum*, which describes a state of the quantum field containing no particles. The Minkowski vacuum is also fundamental for the perturbative or non-perturbative construction of interacting theories, corresponding to the quantization of non-linear classical field equations.

Quantum Field Theory on Minkowski spacetime relies heavily on its symmetry under the Poincaré group. This is apparent in the ubiquitous role of plane waves in the analysis of classical field equations, but more importantly in the characterization of the Minkowski vacuum as the unique state which is invariant under the Poincaré group and has some *energy positivity* property.

Quantum Field Theory on *curved spacetimes* describes quantum fields in an external gravitational field, represented by the Lorentzian metric of the ambient spacetime. It is used in situations when both the quantum nature of the fields and the effect of gravitation are important, but the quantum nature of gravity can be neglected in a first approximation. Its non-relativistic analog would be for example ordinary Quantum Mechanics, i.e. the Schrödinger equation, in a classical exterior electromagnetic field.

Its most important areas of application are the study of phenomena occurring in the early universe and in the vicinity of black holes, and its most celebrated result is the discovery by Hawking that quantum particles are created near the horizon of a black hole.

The symmetries of the Minkowski spacetime, which play such a fundamental role, are absent in curved spacetimes, except in some simple situations, like *stationary* or *static* spacetimes. Therefore, the traditional approach to quantum field theory has to be modified: one has first to perform an *algebraic quantization*, which for free theories amounts to introducing an appropriate *phase space*, which is either a *symplectic* or an *Euclidean* space, in the *bosonic* or *fermionic* case. From such a phase space one can construct CCR or CAR **-algebras*, and actually *nets* of **-algebras*, each associated to a region of spacetime.

The second step consists in singling out, among the many states on these **-algebras*, the physically meaningful ones, which should resemble the Minkowski vacuum, at least in the vicinity of any point of the spacetime. This leads to the notion of *Hadamard states*, which were originally defined by requiring that their two-point functions have a specific asymptotic expansion near the diagonal, called the *Hadamard expansion*.

A very important progress was made by Radzikowski, [R1, R2], who introduced the characterization of Hadamard states by the *wavefront set* of their two-point functions. The wavefront set of a distribution is the natural way to describe its singularities in the cotangent space, and lies at the basis of *microlocal analysis*, a fundamental tool in the analysis of linear and non-linear partial differential equations. Among its avatars in the physics literature are, for example, the geometrical optics in wave propagation and the semi-classical limit in Quantum Mechanics.

The introduction of microlocal analysis in quantum field theory on curved spacetimes started a period of rapid progress, not only for free (i.e. linear) quantum fields, but also for the perturbative construction of interacting fields by Brunetti and Fredenhagen [BF]. For free fields it allowed to use several fundamental results of microlocal analysis, like Hörmander's propagation of singularities theorem and the classification of parametrices for Klein–Gordon operators by Duistermaat and Hörmander.

1.2 Content

The goal of these lecture notes is to give an exposition of microlocal analysis methods in the study of Quantum Field Theory on curved spacetimes. We will focus on *free fields* and the corresponding *quasi-free states* and more precisely on *Klein–Gordon fields*, obtained by quantization of linear Klein–Gordon equations on Lorentzian manifolds, although the case of *Dirac fields* will be described in Chapter 17.

There exist already several good textbooks or lecture notes on quantum field theory in curved spacetimes. Among them let us mention the book by Bär, Ginoux and Pfäeffle [BGP], the lecture notes [BFr] and [BDFY], the more recent book by Rejzner [Re], and the survey by Benini, Dappiagi and Hack [BDH]. There exist also more physics oriented books, like the books by Wald [W2], Fulling [F] and Birrell and Davies [BD]. Several of these texts contain important developments which are not described here, like the perturbative approach to interacting theories, or the use of category theory.

In this lecture notes we focus on advanced methods from microlocal analysis, like for example *pseudodifferential calculus*, which turn out to be very useful in the study and construction of Hadamard states.

Pure mathematicians working in partial differential equations are often deterred by the traditional formalism of quantum field theory found in physics textbooks, and by the fact that the construction of interacting theories is, at least for the time being, restricted to perturbative methods.

We hope that these lecture notes will convince them that quantum field theory on curved spacetimes is full of interesting and physically important problems, with a nice interplay between algebraic methods, Lorentzian geometry and microlocal methods in partial differential equations. On the other hand, mathematical physicists with a traditional education, which may lack familiarity with more advanced tools of microlocal analysis, can use this text as an introduction and motivation to the use of these methods.

Let us now give a more detailed description of these lecture notes. The reader may also consult the introduction of each chapter for more information.

For pedagogical reasons, we have chosen to devote Chapters 2 and 3 to a brief outline of the traditional approach to quantization of Klein–Gordon fields on Minkowski spacetime, but the impatient reader can skip them without trouble.

Chapter 4 deals with CCR $*$ -algebras and quasi-free states. A reader with a PDE background may find the reading of this chapter a bit tedious. Nevertheless, we think it is worth the effort to get familiar with the notions introduced there.

In Chapter 5 we describe well-known concepts and results concerning Lorentzian manifolds and Klein–Gordon equations on them. The most important are the notion of *global hyperbolicity*, a property of a Lorentzian manifold implying global solvability of the Cauchy problem, and the *causal propagator* and the various symplectic spaces associated to it.

In Chapter 6 we discuss quasi-free states for Klein–Gordon fields on curved spacetimes, which is a concrete application of the abstract formalism in Chapter 4. Of interest are the two possible descriptions of a quasi-free state, either by its spacetime covariances, or by its Cauchy surface covariances, which are both important in practice. Another useful point is the discussion of conformal transformations.

Chapter 7 is devoted to the microlocal analysis of Klein–Gordon equations. We collect here various well-known results about wavefront sets, Hörmander’s propagation of singularities theorem and its related study with Duistermaat of *distinguished parametrices* for Klein–Gordon operators, which play a fundamental role in quantized Klein–Gordon fields.

In Chapter 8 we introduce the modern definition of Hadamard states due to Radzikowski and discuss some of its consequences. We explain the equivalence with the older definition based on Hadamard expansions and the well-known existence result by Fulling, Narcowich and Wald.

In Chapter 9 we discuss ground states and thermal states, first in an abstract setting, then for Klein–Gordon operators on stationary spacetimes. Ground states share the symmetries of the background stationary spacetime and are the natural analogs of the Minkowski vacuum. In particular, they are the simplest examples of Hadamard states.

Chapter 10 is devoted to an exposition of a global pseudodifferential calculus on non compact manifolds, the *Shubin calculus*. This calculus is based on the notion of manifolds of bounded geometry and is a natural generalization of the standard uniform calculus on \mathbb{R}^n . Its most important properties are the Seeley and Egorov theorems.

In Chapter 11 we explain the construction of Hadamard states using the pseudodifferential calculus in Chapter 10. The construction is done, after choosing a Cauchy surface, by a microlocal splitting of the space of Cauchy data obtained from a global construction of parametrices for the Cauchy problem. It can be applied to many spacetimes of physical interest, like the Kerr–Kruskal and Kerr–de Sitter spacetimes.

In Chapter 12 we construct *analytic* Hadamard states by *Wick rotation*, a well-known procedure in Minkowski spacetime. Analytic Hadamard states are defined on analytic spacetimes, by replacing the usual C^∞ wavefront set by the *analytic*

wavefront set, which describes the analytic singularities of distributions. Like the Minkowski vacuum, they have the important *Reeh–Schlieder* property. After Wick rotation, the hyperbolic Klein–Gordon operator becomes an elliptic Laplace operator, and analytic Hadamard states are constructed using a well-known tool from elliptic boundary value problems, namely the *Calderón projector*.

In Chapter 13 we describe the construction of Hadamard states by the *characteristic Cauchy problem*. This amounts to replacing the space-like Cauchy surface in Chapter 11 by a past or future *lightcone*, choosing its interior as the ambient spacetime. From the trace of solutions on this cone one can introduce a *boundary symplectic space*, and it turns out that it is quite easy to characterize states on this symplectic space which generate a Hadamard state in the interior. Its main application is the *conformal wave equation* on spacetimes which are asymptotically flat at past or future *null infinity*. We also describe in this chapter the *BMS group* of asymptotic symmetries of these spacetimes, and its relationship with Hadamard states.

In Chapter 14 we discuss Klein–Gordon fields on spacetimes with Killing horizons. Our aim is to explain a phenomenon loosely related with the *Hawking radiation*, namely the existence of the *Hartle–Hawking–Israel* vacuum, on spacetimes having a stationary Killing horizon. The construction and properties of this state follow from the Wick rotation method already used in Chapter 12, the Calderón projectors playing also an important role.

Chapter 15 is devoted to the construction of Hadamard states by scattering theory methods. We consider spacetimes which are asymptotically static at past or future time infinity. In this case one can define the *in* and *out vacuum states*, which are states asymptotic to the vacuum state at past or future time infinity. Using the tools from Chapters 10, 11 we prove that these states are Hadamard states.

In Chapter 16 we discuss the notion of *Feynman inverses*. It is known that a Klein–Gordon operator on a globally hyperbolic spacetime admits *Feynman parametrices*, which are unique modulo smoothing operators and characterized by the wavefront set of its distributional kernels. One can ask if one can also define a unique, canonical true inverse, having the correct wavefront set. We give a positive answer to this question on spacetimes which are *asymptotically Minkowski*.

Chapter 17 is devoted to the quantization of the *Dirac equation* and to the definition of Hadamard states for Dirac quantum fields. The Dirac equation on a curved spacetime describes an electron-positron field which is a *fermionic field*, and the CCR $*$ -algebra for the Klein–Gordon field has to be replaced by a CAR $*$ -algebra. Apart from this difference, the theory for fermionic fields is quite parallel to the bosonic case. We also describe the quantization of the *Weyl equation*, which originally was thought to describe massless neutrinos.

1.2.1 Acknowledgments. The results described in Chapters 11, 12, 15, and part of those in Chapters 10 and 13, originate from common work with Michal Wrochna, over a period of several years.

I learned a lot of what I know about quantum field theory from my long collaboration with Jan Dereziński, and several parts of these lecture notes, like Chapters 4

and 5 borrow a lot from our common book [DG]. I take the occasion here to express my gratitude to him.

Finally, I also greatly profited from discussions with members of the AQFT community. Among them I would like to especially thank Claudio Dappiagi, Valter Moretti, Nicola Pinamonti, Igor Khavkine, Klaus Fredenhagen, Detlev Buchholz, Wojciech Dybalski, Kasia Rejzner, Dorothea Bahns, Rainer Verch, Stefan Hollands and Ko Sanders.

1.3 Notation

We now collect some notation that we will use.

We set $\langle \lambda \rangle = (1 + \lambda^2)^{\frac{1}{2}}$ for $\lambda \in \mathbb{R}$.

We write $A \Subset B$ if A is relatively compact in B .

If X, Y are sets and $f : X \rightarrow Y$ we write $f : X \xrightarrow{\sim} Y$ if f is bijective. If X, Y are equipped with topologies, we write $f : X \rightarrow Y$ if the map is continuous, and $f : X \xrightarrow{\sim} Y$ if it is a homeomorphism.

1.3.1 Scale of abstract Sobolev spaces. Let \mathcal{H} a real or complex Hilbert space and A a selfadjoint operator on \mathcal{H} . We write $A > 0$ if $A \geq 0$ and $\text{Ker } A = \{0\}$.

If $A > 0$ and $s \in \mathbb{R}$, we equip $\text{Dom } A^{-s}$ with the scalar product $(u|v)_{-s} = (A^{-s}u|A^{-s}v)$ and the norm $\|A^{-s}u\|$. We denote by $A^s\mathcal{H}$ the completion of $\text{Dom } A^{-s}$ for this norm, which is a (real or complex) Hilbert space.