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From the Vlasov-Maxwell-Boltzmann system to incompressible viscous electro-magneto-hydrodynamics. Volume 1. (English)

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The book is devoted to the analysis of viscous hydrodynamics limits, mainly from a fundamental point of view. A second book will indeed follow the present one with applications. The present book is divided into two parts and twelve chapters. Three appendices complete the book.
The first part contains three chapters, devoted to the presentation of Vlasov-Maxwell-Boltzmann system and its properties, and to the corresponding mathematical framework. It gathers weak stability results for the limiting macroscopic systems. The second part presents the formal limits which are deduced from this Vlasov-Maxwell-Boltzmann system according to different scalings.
Chapter 1 starts with the Vlasov-Maxwell-Boltzmann system which describes viscous incompressible magneto-hydrodynamics. It is written as $\partial_t f + v\cdot\nabla_x f + \frac{q}{m}(E+v\wedge B)\cdot\nabla_v f = Q(f,f)$, $\mu_0\epsilon_0\partial_t E - \mathrm{rot}B = -\mu_0 q\int_{\mathbb{R}^3} fv\,dv$, $\partial_t B + \mathrm{rot}E = 0$, $\operatorname{div} E = \frac{q}{\epsilon_0}(\int_{\mathbb{R}^3} f dv - 1)$, $\operatorname{div} B = 0$, where f is the density of charged particles which depends on the time $t\in(0,\infty)$, on the position x and on the velocity v, q is the charge, m is the mass, E is the electric field, B is the magnetic field, Q is Boltzmann collision operator, μ_0 is the vacuum permeability and ϵ_0 is the vacuum permittivity. Because the charged ions may be positive or negative, the Vlasov-Boltzmann equation may be split in two equations $\partial_t f^+ + v\cdot\nabla_x f^+ + \frac{q^+}{m^+}(E+v\wedge B)\cdot\nabla_v f^+ = Q(f^+,f^+)+Q(f^+,f^-)$ for the cations and $\partial_t f^- + v\cdot\nabla_x f^- - \frac{q^-}{m^-}(E+v\wedge B)\cdot\nabla_v f^- = Q(f^-,f^-)+Q(f^-,f^+)$ for the anions, the integral $\int_{\mathbb{R}^3} qfv\,dv$ being replaced in the Ampère and Gauss equations by $\int_{\mathbb{R}^3}(q^+f^+ - q^-f^-)v\,dv$ or $\int_{\mathbb{R}^3}(q^+f^+ - q^-f^-)v\,dv$, respectively. The Boltzmann collision operator is taken as $Q(f,h) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2}(f(v')h(v'_*) - f(v)h(v_*))b(v-v_*,\sigma)\,d\sigma\,dv_*$ where $v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma$, $v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma$. The collision kernel b depends on the relative velocity $|z|$ and on the deviation angle θ as $b(|z|,\cos\theta) = \frac{\beta}{\sin\epsilon}\frac{\partial\beta}{\partial\theta}|z|$, where β is the impact parameter. The authors do not assume the usual simplifying cutoff hypothesis for the collision kernel. The authors then derive that the entropy dissipation is nonnegative, which leads to Boltzmann H-theorem and that the entropy $\int_{\mathbb{R}^3} f\log f dv$ is at least formally a Lyapunov function for the Boltzmann equation. They also prove the entropy inequality. The authors finally explain that they will consider renormalized solutions to the Vlasov-Maxwell-Boltzmann system although their existence is not yet proved. They indeed explain the difficulty to prove the convergence to renormalized solutions of approximate solutions to the Vlasov-Maxwell-Boltzmann system. Such approximate solutions instead converge to measure-valued renormalized solutions with the introduction of Young measures.
Chapter 2 starts with the description of incompressible viscous regimes. The authors write the Boltzmann equation in adimensional variables as $S_t\partial_t f + v\cdot\nabla_x f = \frac{1}{\mathrm{Kn}}Q(f,f)$, where Kn is the Knudsen number equal to the ratio λ_0/l_0 between the mean free path λ_0 and the observation length scale l_0, St is the Strouhal number equal to l_0/c_0t_0 with c_0 being the speed of sound and t_0 being the unit time. The authors also introduce the Mach number as $Ma = u_0/c_0$ where u_0 is the bulk velocity. They consider hydrodynamics limits obtained when Kn goes to 0 (hence $Ma\to 0$) and they take the density f in the form $f = M(1+\mathrm{Ma}g)$ where M is the global normalized Maxwellian equilibrium of density 1, bulk velocity 0 and temperature 1 defined as $M(v) = \frac{1}{(2\pi)^{\frac{3}{2}}}e^{-\frac{|v|^2}{2}}$ and g is a fluctuation. They rewrite the Vlasov-Maxwell-Boltzmann system, assuming that Kn, St and Ma are of the same order ϵ.

A parameter δ appears which has to be compared to this order ϵ. Changing also the units of E and B, the authors get the problem $\epsilon\partial_t f + v\cdot\nabla_x f + (\alpha E + \beta v\wedge B)\cdot\nabla_v f = \frac{1}{\epsilon}Q(f,f)$, $f = M(1+\epsilon g)$, $\gamma\partial_t E - \mathrm{rot} B = -\frac{\beta}{\epsilon^2}\int_{\mathbb{R}^3} f v\,dv$, $\gamma\partial_t B + \mathrm{rot} E = 0$, $\mathrm{div}\, E = \frac{\alpha}{\epsilon^2}(\int_{\mathbb{R}^3} f\,dv - 1)$, $\mathrm{div}\, B = 0$, for one species and $\epsilon\partial_t f^{\pm} + v\cdot\nabla_x f^{\pm} \pm (\alpha E + \beta v\wedge B)\cdot\nabla_v f^{\pm} = \frac{1}{\epsilon}Q(f^{\pm},f^{\pm}) + \frac{\delta}{\epsilon^2}Q(f^{\pm},f^{\mp})$, $f^{\pm} = M(1+\epsilon g^{\pm})$, $\gamma\partial_t E - \mathrm{rot} B = -\frac{\beta}{\epsilon^2}\int_{\mathbb{R}^3}(f^+ - f^-)v\,dv$, $\gamma\partial_t B + \mathrm{rot} E = 0$, $\mathrm{div}\, E = \frac{\alpha}{\epsilon^2}(\int_{\mathbb{R}^3}(f^+ - f^-)\,dv - 1)$, $\mathrm{div}\, B = 0$ for two species. They formally derive the asymptotic systems according to the orders of this parameter δ. They conclude Chapter 2 with the formal derivations of the asymptotic systems in the case of two species.

In Chapter 3, the authors analyze the well-posedness of three asymptotic problems which have been formally derived in Chapter 2: an incompressible quasi-static Navier-Stokes-Fourier-Maxwell-Poisson system, the two-fluid incompressible Navier-Stokes-Fourier-Maxwell system with Ohm's law and the two-fluid incompressible Navier-Stokes-Fourier-Maxwell system with solenoidal Ohm's law. The chapter starts with the incompressible quasi-static Navier-Stokes-Fourier-Maxwell-Poisson system, written as $\partial_t u + u\cdot\nabla_x u = -\nabla_x p + E + \rho\nabla_x\theta + u\wedge B$, $\mathrm{div}\, u = 0$, $\partial_t(\frac{3}{2}\theta - \rho) + u\cdot\nabla_x(\frac{3}{2}\theta - \rho) - \frac{5}{2}\kappa\Delta_x\theta = 0$, $\Delta_x(\rho+\theta) = \rho$, $rot B = u$, $\mathrm{div}\, E = \rho$ $\partial_t B + rot E = 0$, $\mathrm{div}\, B = 0$. Initial conditions $(\rho^{\mathrm{in}}, u^{\mathrm{in}}, \theta^{\mathrm{in}}, B^{\mathrm{in}})$ are added. The authors first prove a global energy inequality, assuming that (ρ, u, θ, B) is a smooth solution to this problem. They define the notion of weak or Leray solution, and they prove the existence of such a weak solution under hypotheses on the initial data. They then move to the Navier-Stokes-Fourier-Maxwell system with Ohm's law and the two-fluid incompressible Navier-Stokes-Fourier-Maxwell system with solenoidal Ohm's law. They prove that a smooth solution (u, E, B) to these problems satisfies a global conservation of energy. They quote from the literature the existence of large global solutions in the 2D case to the Navier-Stokes-Maxwell system, or a local small solution to this problem in the 3D case. Chapter 3 ends with the proof of weak-strong stability and existence results for dissipative solutions to these two systems.

Part II begins with Chapter 4 which is devoted to a deeper analysis of two asymptotic systems. The authors first define the notion of renormalized solutions to the Vlasov-Maxwell-Boltzmann system with one or two species, first for the Vlasov-Boltzmann equation $\partial_t f + v\cdot\nabla_x f + F\cdot\nabla_v f = Q(f,f)$, assuming that the given force F satisfies at least $F, \nabla_v\cdot F \in L^1_{\mathrm{loc}}(dtdx; L^1(M^\alpha dv))$ for all $\alpha > 0$, where M is the global asymptotic Maxwellian equilibrium. Assuming now classical hypotheses on the kernel b, on F and on the initial data, the authors recall the existence of a renormalized solution to the Vlasov-Boltzmann equation which further satisfies a local conservation of mass property and the global entropy inequality. The authors then define the notion of a renormalized solution to the Vlasov-Maxwell-Boltzmann system with one or two species and they analyze macroscopic conservation laws within this context. In the case of the Navier-Stokes-Fourier-Maxwell-Poisson system, the authors assume the existence of a renormalized solution to the one species Vlasov-Maxwell-Boltzmann system and they prove a weak relatively compactness result in $L^1_{\mathrm{loc}}(dtdx)$ for special macroscopic fluctuations of the density, bulk velocity and temperature and under hypotheses on the initial data. The limit is a weak solution to the Navier-Stokes-Fourier-Maxwell-Poisson system. The proof of this result is postponed to Chapter 11. In the case of the Navier-Stokes-Fourier-Maxwell system with solenoidal Ohm's law, the authors also assume the existence of renormalized solutions to the Vlasov-Maxwell-Boltzmann system with two species. They again prove a weak relatively compactness result is $L^1_{\mathrm{loc}}(dtdx)$ and that the limit is a non-positive solution to the Navier-Stokes-Fourier-Maxwell system with solenoidal Ohm's law. They here assume that the initial data satisfy different hypotheses, among which a "well-prepared" hypothesis. A quite similar weak relatively compactness result is proved in the case of Navier-Stokes-Fourier-Maxwell system with Ohm's law, and the proofs of these results are postponed to Chapter 12.

In Chapter 5, the authors prove weak compactness results for fluctuations in the case of the Vlasov-Boltzmann equation or the Vlasov-Maxwell-Boltzmann system with one species. The authors first derive the relation between the entropy bounds and the entropy dissipative bounds. They present a decomposition of the linearized collision operator and properties of the linear collision operator. The chapter ends with improved integrability results on the velocity for the fluctuations.

In Chapter 6, the authors derive lower-order linear macroscopic constraint equations, using weak compactness methods, first for one species, then for two species with weak interactions. They prove energy estimates for sequences of renormalized solutions with one or two species to the Vlasov-Maxwell-Boltzmann system. The chapter ends with considerations on the difficulty to pass to the limit in Maxwell equations.

In Chapter 7, the authors prove strong compactness results for the fluctuations. They first assume that the collision operator b is smooth and compactly supported, and they derive a compactness result with respect to the velocity v. They quote from previous results they obtained, locally relatively compact results in $L^p(\mathbb{R}_t \times \mathbb{R}^3_x \times \mathbb{R}^3_v)$, $1 < p < \infty$, for families of functions in this space which satisfy a locally relatively compactness property with respect to v and further properties which allow to use the hypoellipticity property of the free transport equation. The chapter ends with the proof of compactness results for fluctuations in the one or two species cases.

In Chapter 8, the authors consider the case with two species. Using the symmetries of the collision integrands $q^{\pm}$ and $q^{\mp}$, they derive a singular limit for a sequence of renormalized solutions to the scaled Vlasov-Maxwell-Boltzmann system in the case of weak interspecies interactions. They assume that $\delta = o(1)$, δ/ϵ is unbounded, and different conditions on the data of the problem are imposed. They then characterize the limiting kinetic equations for the case of two species for strong interspecies interactions, assuming that $\delta = 1$ and different conditions on the data. They also characterize the limiting collision integrands and, finally, the limiting energy inequality in this case.

Chapter 9 investigates the consistency of the electro-magneto-hydrodynamic approximation for the incompressible quasi-static Navier-Stokes-Fourier-Maxwell-Poisson system. The authors consider the admissible nonlinearity $\Gamma(z) - 1 = (z-1)\gamma(z)$ with $\gamma \in C^1([0,\infty);\mathbb{R})$ satisfying further conditions, and they first build approximate conservation laws for the scaled one-species Vlasov-Maxwell-Boltzmann system, considering fluctuations of the kind $g_\epsilon \gamma_\epsilon \chi(\frac{|v|^2}{K|\log \epsilon|})$ where $\chi \in C^\infty_c([0,\infty))$ with $1_{[0,1]} \le \chi \le 1_{[0,2]}$ and $K > 0$ is large enough. They prove that the associated reminders converge to 0 in $L^1_{\mathrm{loc}}(dt; W^{-1,1}_{\mathrm{loc}}(dx))$. They also prove that the conservation defects converge to 0 in $L^1_{\mathrm{loc}}(dtdx)$. They finally build approximate conservation of mass, momentum and energy in the case of two species proving estimates and convergence to 0 on the reminders.

In Chapter 10, the authors consider the case of one species and they analyze the time oscillations in the incompressible quasi-static Navier-Stokes-Fourier-Maxwell-Poisson system which allows to derive the weak stability and the convergence of the Vlasov-Maxwell-Boltzmann system as ϵ tends to 0. They introduce the singular linear system $\partial_t \begin{pmatrix} \rho_\epsilon \\ u_\epsilon \\ \sqrt{\frac{3}{2}}\theta_\epsilon \\ E_\epsilon \\ B_\epsilon \end{pmatrix} + \frac{1}{\epsilon} W \begin{pmatrix} \rho_\epsilon \\ u_\epsilon \\ \sqrt{\frac{3}{2}}\theta_\epsilon \\ E_\epsilon \\ B_\epsilon \end{pmatrix} = O(1)$, where $W : L^2(dx) \to H^{-1}(dx)$ is defined through $W = \begin{pmatrix} 0 & \mathrm{div} & 0 & 0 & 0 \\ \nabla_x & 0 & \sqrt{\frac{2}{3}}\nabla_x & -\mathrm{Id} & 0 \\ 0 & \sqrt{\frac{2}{3}}\,\mathrm{div} & 0 & 0 & 0 \\ 0 & Id & 0 & 0 & -\mathrm{rot} \\ 0 & 0 & 0 & \mathrm{rot} & 0 \end{pmatrix}$. They also introduce the Leray projector $P : L^2(dx) \to L^2(dx)$ onto solenoidal vector fields. The main result of this chapter proves a weak stability result for acoustic and electromagnetic waves in the sense of distributions for nonlinear terms using the Leray projector, considering a sequence of renormalized solutions to the scaled one-species Vlasov-Maxwell-Boltzmann system.

Chapter 11 is devoted to the proof of the result announced in Chapter 4 and concerning the convergence of the renormalized solution to the one-species Vlasov-Maxwell-Boltzmann system to a weak solution to the incompressible quasi-static Navier-Stokes-Fourier-Maxwell-Poisson system.

In the final Chapter 12, the authors consider the asymptotics leading to the two-fluid incompressible Navier-Stokes-Fourier-Maxwell system. The main tool is a relative entropy method. They prove the results announced in Chapter 4, first for weak interactions and finally for strong interactions.

Appendix A gives a short analysis of the cross-section for momentum and energy transfer. Appendix B is devoted to the presentation and the properties of Young measures. Appendix C gives short complements to the hypoelliptic transfer of compactness which has been used in Chapter 7.

The book presents deep results concerning the Vlasov-Maxwell-Boltzmann system and its asymptotics.

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