# Prologue

The subject of this monograph is the shock development problem in fluid mechanics. This problem is formulated in the framework of the Eulerian equations of a compressible perfect fluid as completed by the laws of thermodynamics. These equations express the differential conservation laws of mass, momentum, and energy and constitute a quasilinear hyperbolic 1st-order system for the physical variables, that is, the fluid velocity and the two positive quantities corresponding to a local thermodynamic equilibrium state. Smooth initial data for this system of equations lead to the formation of a surface in spacetime where the derivatives of the physical quantities with respect to the standard rectangular coordinates blow up. Now, there is a mathematical notion of maximal development of initial data. As was first shown in the monograph [Ch-S], this maximal development ends at a future boundary which consists of a regular part $\underline{\mathcal{C}}$ and a singular part $\mathcal{B}$ with a common past boundary $\partial_{-}\mathcal{B}$, the surface just mentioned. A solution of the Eulerian equations in a given spacetime domain defines a cone field on this domain, that is, a cone in the tangent space at each point in the domain, the field of sound cones. This defines a causal structure on the spacetime domain, equivalent to a conformal class of Lorentzian metrics, the acoustical causal structure. Relative to this structure $\partial_{-}\mathcal{B}$ is a spacelike surface, while $\underline{\mathcal{C}}$ is a null hypersurface. Also $\mathcal{B}$ is in this sense a null hypersurface; however, being singular, while its intrinsic geometry is that of a null hypersurface, its extrinsic geometry is that of a spacelike hypersurface, because the past null geodesic cone in the spacetime manifold of a point on $\mathcal{B}$ does not intersect $\mathcal{B}$. The character of $\mathcal{B}$ and the behavior of the solution at $\mathcal{B}$ were described in detail in [Ch-S] by means of the introduction of a class of coordinates such that the rectangular coordinates as well as the physical variables are smooth functions of the new coordinates up to $\mathcal{B}$, but the Jacobian of the transformation to the new coordinates, while strictly negative in the past of $\mathcal{B}$, vanishes at $\mathcal{B}$ itself, a fact which characterizes $\mathcal{B}$. Now, the mathematical notion of maximal development of initial data, while physically correct up to $\underline{\mathcal{C}} \bigcup \partial_{-}\mathcal{B}$, is not physically correct up to $\mathcal{B}$. The problem of the physical continuation of the solution is set up in the epilogue of [Ch-S] as the shock development problem. In this problem, one is required to construct a hypersurface of discontinuity $\mathcal{K}$, the shock hypersurface, lying in the past of $\mathcal{B}$ but having the same past boundary as the latter, namely $\partial_{-}\mathcal{B}$, and a solution of the Eulerian equations in the spacetime domain bounded in the past by $\underline{\mathcal{C}} \bigcup \mathcal{K}$, agreeing on $\underline{\mathcal{C}}$ with the data induced by the maximal development, while having jumps across $\mathcal{K}$ relative to the data induced on $\mathcal{K}$ by the maximal development. These jumps satisfy the jump conditions which follow from the integral form of the mass, momentum, and energy conservation laws, that is, by requiring the corresponding differential laws to hold in the weak sense in a full neighborhood of $\mathcal{K}$. Moreover, $\mathcal{K}$ is required to be a spacelike hypersurface relative to the acoustical structure corresponding to the prior solution and a timelike hypersurface relative to the acoustical structure corresponding to the new solution (see Sections 1.4, 1.5 and Figure 1.3). The 1st requirement implies that the prior solution along $\mathcal{K}$ is given by the maximal development discussed above, while the 2nd requirement is what gives the problem the character of a free boundary problem, the jump conditions at $\mathcal{K}$ being just right to determine both the location of the shock hypersurface in the underlying background spacetime, as well as the new solution. Thus, the singular surface $\partial_{-}\mathcal{B}$ is the cause generating the shock hypersurface $\mathcal{K}$. In view of the acoustically spacelike nature of $\mathcal{K}$ relative to the prior solution, the shock hypersurface penetrates the interior of the domain of the maximal development rendering unphysical the part of the maximal development lying to the future of $\mathcal{K}$. We note that the results of the previous monograph [Ch-S] are to be used in order to set up the initial data on $\underline{\mathcal{C}}$ as well as for providing the data on the past side of $\mathcal{K}$. However, only the qualitative features of the prior maximal development will be used and not any smallness conditions, as would follow from the assumption in [Ch-S] that the initial data from which this maximal development arises correspond to a small departure from those of a constant state.

The monograph [Ch-S] actually considered the extension of the Eulerian equations to the framework of special relativity. In this framework the underlying geometric structure of the spacetime manifold is that of the Minkowski spacetime of special relativity. On the other hand, the underlying spacetime structure of the original Eulerian equations, as of all of classical mechanics, is that of Galilei spacetime. The monograph [Ch-Mi] treated the same topics as [Ch-S] reaching similar results in a considerably simpler, self-contained manner. The relationship of the non-relativistic to the relativistic theory, how results in the former are deduced as limits of results in the latter, is discussed in detail in Sections 1.3–1.6. The Galilean structure has a distinguished family of hyperplanes, those of absolute simultaneity, while the Minkowski structure is that of a flat Lorentzian manifold; thus the latter, with its light cone field, is in a sense more like that corresponding to the sound cone field defined by a solution of the Eulerian equations. What we mean by rectangular coordinates in the Minkowskian framework is the standard geometric notion and two such systems of coordinates are related by a transformation belonging to the Poincaré group. On the other hand, by rectangular coordinates in the Galilean framework we mean a Galilei frame together with rectangular coordinates in Euclidean space and two such systems of coordinates are related by the Galilei group, which extends the Euclidean group. The simplest representative in the conformal class of Lorentzian metrics corresponding to the sound cone field is the *acoustical metric*. This is given in the relativistic theory in terms of rectangular coordinates by

$$h = h_{\mu\nu} dx^{\mu} \otimes dx^{\nu}, \quad h_{\mu\nu} = g_{\mu\nu} + (1 - \eta^2) u_{\mu} u_{\nu}, \quad u_{\mu} = g_{\mu\nu} u^{\nu}, \tag{P.1}$$

where $g_{\mu\nu}$ are the components of the Minkowski metric, $u^{\mu}$ the components of the spacetime fluid velocity $u$, a future-directed vectorfield which is timelike and of unit magnitude with respect to the Minkowski metric, and $\eta$ is the sound speed, a thermodynamic function. In the non-relativistic theory the acoustical metric is given in terms of rectangular coordinates by

$$h = -\eta^2 dt \otimes dt + (dx^i - v^i dt) \otimes (dx^i - v^i dt) = h_{\mu\nu} dx^{\mu} dx^{\nu}, \quad x^0 = t. \tag{P.2}$$

Here $\eta$ is again the sound speed while $v^i$ are the components of the spatial fluid velocity, the spacetime fluid velocity $u$ being given in the non-relativistic theory by

$$u = \frac{\partial}{\partial t} + v^i \frac{\partial}{\partial x^i}. \tag{P.3}$$

In (P.1), (P.2), (P.3), as in the entire monograph, we follow the summation convention according to which repeated indices are meant summed over their range.

From the mathematical point of view the shock development problem is a free boundary problem, with nonlinear conditions at the free boundary $\mathcal{K}$, for a quasilinear hyperbolic 1st-order system, with characteristic initial data on $\underline{\mathcal{C}}$ which are singular, in a prescribed manner, at $\partial_{-}\mathcal{B}$, the past boundary of $\underline{\mathcal{C}}$. It will be shown that the singularity persists, not only as a discontinuity in the physical variables across $\mathcal{K}$, but also as a milder singularity propagating along $\underline{\mathcal{C}}$. While the physical variables and their 1st derivatives extend continuously across $\underline{\mathcal{C}}$, the 1st derivatives are only $C^{0,1/2}$ at $\underline{\mathcal{C}}$ from the point of view of the future solution. Majda's pioneering works [Ma1], [Ma2] solved the local shock continuation problem. This is the problem of continuing locally in time a solution displaying a shock discontinuity initially. It is a problem of the same kind, but without the singular initial conditions of the shock development problem which signal that a shock is about to form. The spherically symmetric barotropic case of the shock development problem was recently solved in [Ch-Li].

What the present monograph solves is not the general shock development problem but what we call the restricted shock development problem. This problem is formulated in Section 1.6 and results if we disregard the Hugoniot relation, one of the jump conditions which constitutes a relation between thermodynamic quantities on the two sides of $\mathcal{K}$ (see (1.255) for the relativistic relation, (1.310) for the non-relativistic relation). This relation expresses $\triangle s$, the jump in entropy, in terms of $\triangle p$, the jump in pressure, giving $\triangle s$ as proportional to $(\triangle p)^3$ to leading terms (see (1.260) for the relativistic expression, (1.318) for the non-relativistic expression).

Then disregarding the Hugoniot relation, we set $\triangle s = 0$ to obtain the restricted problem. Due to the fact that in terms of the coordinate $\tau$ on $\mathcal{K}$, which is introduced in Section 2.5 and which vanishes at $\partial_{-}\mathcal{B}$, $\triangle p \sim \tau$, reinstating the Hugoniot relation would give $\triangle s \sim \tau^3$; consequently the restricted problem retains the main difficulty of the general problem, namely the singular behavior as $\tau \to 0$. The jump conditions other than the Hugoniot relation are fully taken into account in the framework of the restricted shock development problem. The corresponding restricted shock continuation problem was solved by Majda and Thomann [Ma-Th]. As we shall see, our solution of the restricted shock development problem relies on [Ma-Th], as we obtain our solution via a regularization of the initial conditions and an application of the method of continuity which naturally must rely on a local existence theorem for regular initial conditions.

Our treatment is based on the 1-form $\beta$, a general concept, given in the relativistic theory (see Section 1.1) in terms of rectangular coordinates by

$$\beta = \beta_\mu dx^\mu, \quad \beta_\mu = -hu_\mu, \tag{P.4}$$

where $h$ is the relativistic enthalpy, a thermodynamic function. In the non-relativistic theory (see Section 1.3), the 1-form $\beta$ is given in terms of rectangular coordinates by

$$\beta = \left(h + \frac{1}{2}|v|^2\right) dt - v^i dx^i = \beta_\mu dx^\mu, \tag{P.5}$$

where $h$ is the (non-relativistic) enthalpy. Associated to the 1-form $\beta$ is the spacetime vorticity 2-form $\omega = -d\beta$, which satisfies, in both the relativistic and the non-relativistic theories, the equation

$$i_u\omega = \theta\, ds, \tag{P.6}$$

where $u$ is again the spacetime fluid velocity and $\theta$ is the temperature. This equation is supplemented by the differential mass conservation law to obtain the full content of the Eulerian equations. The differential mass conservation law takes, in rectangular coordinates, the form

$$\frac{\partial(nu^\mu)}{\partial x^\mu} = 0 \tag{P.7}$$

in both the relativistic and the non-relativistic theories, $n$ being the mass density, a thermodynamic function. Note that the components $\beta_\mu$ of $\beta$ determine the components $u^\mu$ of $u$ as well as the enthalpy $h$, and vice versa. Taking the enthalpy $h$ and the entropy $s$ as the basic thermodynamic variables, any thermodynamic function, in particular the sound speed $\eta$, is a known function of $h$ and $s$ (the equation of state being given). Thus the $\beta_\mu$ together with $s$ determine the components $h_{\mu\nu}$ of the acoustical metric in both the relativistic and the non-relativistic theories; in fact the latter are known smooth functions of the former.

The jump conditions other than the Hugoniot relation consist of what we call the *linear jump condition* and what we call the *nonlinear jump condition*. The linear jump condition states that

$$\beta_{\|+} = \beta_{\|-}, \tag{P.8}$$

where we denote by subscript $-$ the side of $\mathcal{K}$ corresponding to the prior solution, which holds in the past of $\mathcal{K}$, and by a subscript $+$ the side of $\mathcal{K}$ corresponding to the new solution, which holds in the future of $\mathcal{K}$. We then denote by $\beta_{\|\pm}$ the 1-form $\beta$ induced on $\mathcal{K}$ from the two sides. The nonlinear jump condition follows from the integral mass conservation law, that is, by requiring equation (P.7) to hold in the weak sense in a full neighborhood of $\mathcal{K}$. Denoting by $\omega_{\|\pm}$ the vorticity 2-form induced on $\mathcal{K}$ from the two sides, (P.8) implies

$$\omega_{\|+} = \omega_{\|-}. \tag{P.9}$$

In the case that the prior solution is irrotational in a neighborhood of $\mathcal{K}$, as is the case here, we have

$$\omega_{\|+} = 0. \tag{P.10}$$

Then $\omega_+$ reduces to its component $\omega_{\perp+}$ a 1-form intrinsic to $\mathcal{K}$, given according to (P.6) restricted to $\mathcal{K}$ by

$$\omega_{\perp+} = \frac{\theta_+}{u_{\perp+}} d_{\|} s_+, \tag{P.11}$$

where we denote by $d_{\|}$ the differential of a function on $\mathcal{K}$. Also, $u_{\perp+}$ is the component of $u_+$ along the interior unit normal to $\mathcal{K}$ with respect to the Minkowski metric in the relativistic theory, the instantaneous interior unit normal with respect to the Euclidean metric in the non-relativistic theory (see Section 1.4). Here the prior solution is isentropic, being irrotational (note that by irrotational we mean that the spacetime vorticity vanishes, which by (P.6) implies that the entropy is constant), hence in (P.11) we can replace $d_{\|} s_+$ by $d_{\|} \triangle s$ to obtain

$$\omega_{\perp+} = \frac{\theta_+}{u_{\perp+}} d_{\|} \triangle s. \tag{P.12}$$

In the restricted problem the vanishing of $\triangle s$ implies that also $\omega_{\perp+}$ vanishes, hence $\omega_+ = 0$.

As a consequence of equation (P.6), $\omega$ satisfies the transport equation

$$\mathcal{L}_u \omega = d\theta \wedge ds, \tag{P.13}$$

where the right-hand side, like $\omega$ itself, is of differential order 1, the physical variables themselves being of differential order 0. The vectorfield $u$ then defines in the general problem a characteristic field complementing the sound cone field. The hypersurface generated by the integral curves of $u$ initiating at $\partial_- \mathcal{B}$ divides the domain of the new solution into two subdomains: one bounded by $\underline{\mathcal{C}}$ and the hypersurface in question, and one between the hypersurface and $\mathcal{K}$. In the former domain the solution coincides with that of the restricted problem, which is irrotational, while in the latter domain the spacetime vorticity 2-form $\omega$ does not vanish, $\omega_+$ being determined by $\triangle s$ through (P.12). There is then an additional higher-order discontinuity in $\omega$ across the dividing hypersurface.

We shall now describe in brief the mathematical methods introduced in this monograph, noting the essential points and where the corresponding material is found in the monograph. What follows will serve as a guide for the reader. A central role is played by the reformulation in Chapter 2 of the Eulerian equations in the domain $\mathcal{N}$ of the new solution. First a homeomorphism is defined of this domain onto

$$\mathcal{R}_{\delta,\delta} = R_{\delta,\delta} \times S^{n-1}, \tag{P.14}$$

$n$ being the spatial dimension and

$$R_{\delta,\delta} = \left\{ (\underline{u}, u) \in \mathbb{R}^2 \quad : \ 0 \leq \underline{u} \leq u \leq \delta \right\} \tag{P.15}$$

being a domain in $\mathbb{R}^2$, which represents the range in $\mathcal{N}$ of two functions $\underline{u}$ and $u$, the level sets of which are transversal acoustically null hypersurfaces denoted by $\underline{C}_{\underline{u}}$ and $C_u$ respectively, with $\underline{C}_0 = \underline{\mathcal{C}}$. In this representation the shock hypersurface boundary of $\mathcal{N}$ is

$$\mathcal{K}^\delta = \left\{ (\tau, \tau) \quad : \ \tau \in [0, \delta] \right\} \times S^{n-1} \tag{P.16}$$

and $\partial_- \mathcal{B}$ is

$$\partial_- \mathcal{B} = (0,0) \times S^{n-1} = S_{0,0}. \tag{P.17}$$

We denote by $S_{\underline{u},u}$ the surfaces

$$S_{\underline{u},u} = \underline{C}_{\underline{u}} \bigcap C_u = (\underline{u}, u) \times S^{n-1}. \tag{P.18}$$

In the reformulation of the Eulerian equations the unknowns constitute a triplet $((x^\mu : \mu = 0, \ldots, n),\ b,\ (\beta_\mu : \mu = 0, \ldots, n))$, where the $(x^\mu : \mu = 0, \ldots, n)$ are functions on $R_{\delta,\delta} \times S^{n-1}$ representing rectangular coordinates in the corresponding domain in Galilean spacetime in the non-relativistic theory, in Minkowski spacetime in the relativistic theory, with $x^0 = t$, and the $(\beta_\mu : \mu = 0, \ldots, n)$ are also functions on $R_{\delta,\delta} \times S^{n-1}$ and represent the rectangular components of the 1-form $\beta$. The unknown $b$ is a mapping of $R_{\delta,\delta}$ into the space of vectorfields on $S^{n-1}$. The pair $((x^\mu : \mu = 0, \ldots, n),\ b)$ satisfies the *characteristic system*, a fully nonlinear 1st-order system of partial differential equations. The $(\beta_\mu : \mu = 0, \ldots, n)$ satisfy the *wave system*, a quasilinear 1st-order system of partial differential equations. The characteristic system is coupled to the wave system through the $(h_{\mu\nu} : \mu, \nu = 0, \ldots, n)$, which represent the rectangular components of the acoustical metric, and depend on the $(\beta_\mu : \mu = 0, \ldots, n)$. More precisely, denoting by $\not{d} f$ the differential of a function $f$ on the $S_{\underline{u},u}$, the coupling is through the functions $N^\mu : \mu = 0, \ldots, n$ and $\underline{N}^\mu : \mu = 0, \ldots, n$ defined in terms of the $\not{d} x^\mu : \mu = 0, \ldots, n$ pointwise by the conditions

$$h_{\mu\nu} N^\mu \not{d} x^\nu = 0, \quad h_{\mu\nu} N^\mu N^\nu = 0, \quad N^0 = 1 \tag{P.19}$$

and similarly for the $\underline{N}^\mu$. By reason of the quadratic nature of the 2nd of these conditions, a unique pair $((N^\mu : \mu = 0, \ldots, n), (\underline{N}^\mu : \mu = 0, \ldots, n))$ up to exchange is pointwise defined by $(h_{\mu\nu} : \mu, \nu = 0, \ldots, n)$ and $(\not{d} x^\mu : \mu = 0, \ldots, n)$. The vectorfields $N$, $\underline{N}$ with rectangular components $(N^\mu : \mu = 0, \ldots, n)$, $(\underline{N}^\mu : \mu = 0, \ldots, n)$ are then null normal fields, relative to the acoustical metric $h$, to the surfaces $S_{\underline{u},u}$. The exchange ambiguity is removed by requiring $N$ to be tangential to the $C_u$, $\underline{N}$ to be tangential to the $\underline{C}_{\underline{u}}$. The function

$$c = -\tfrac{1}{2} h_{\mu\nu} N^\mu \underline{N}^\nu \tag{P.20}$$

is then bounded from below by a positive constant. Defining

$$L = \frac{\partial}{\partial \underline{u}} - b, \quad \underline{L} = \frac{\partial}{\partial u} + b \tag{P.21}$$

and

$$\rho = Lt, \quad \underline{\rho} = \underline{L}t, \tag{P.22}$$

the characteristic system is

$$Lx^\mu = \rho N^\mu, \quad \underline{L}x^\mu = \underline{\rho}\underline{N}^\mu \quad : \mu = 0, \ldots, n. \tag{P.23}$$

With the vectorfields $N$, $\underline{N}$ and $L$, $\underline{L}$ as defined above, the characteristic system simply expresses the condition that on the one hand $L$ and $N$ and on the other hand $\underline{L}$ and $\underline{N}$, are collinear. In view of the above definitions of $N$, $\underline{N}$ and of $L$, $\underline{L}$, the characteristic system implies that the hypersurfaces $C_u$ and $\underline{C}_{\underline{u}}$ are null with respect to the acoustical metric

$$h = h_{\mu\nu} dx^\mu \otimes dx^\nu, \tag{P.24}$$

their generators being the integral curves of $L$ and $\underline{L}$ respectively.

What is achieved by the reformulation just described is a regularization of the problem. That is, we are now seeking smooth functions on $R_{\delta,\delta} \times S^{n-1}$ satisfying the coupled system, the initial data themselves being represented by smooth functions on $\underline{C}_0$. The Jacobian of the transformation representing the mapping $((\underline{u}, u), \vartheta) \mapsto (x^\mu((\underline{u}, u), \vartheta) : \mu = 0, \ldots, n)$, $\vartheta \in S^{n-1}$ is of the form

$$\frac{\partial(x^0, x^1, \ldots, x^n)}{\partial(\underline{u}, u, \vartheta^1, \ldots, \vartheta^{n-1})} = \rho\underline{\rho} d, \tag{P.25}$$

$(\vartheta^A : A = 1, \ldots, n-1)$ being local coordinates on $S^{n-1}$. Here $\rho$, $\underline{\rho}$ are non-negative functions, defined by (P.22), the inverse temporal density of the foliation of spacetime by the $\underline{C}_{\underline{u}}$ as measured along the generators of the $C_u$, the inverse temporal density of the foliation of spacetime by the $C_u$ as measured along the generators of the $\underline{C}_{\underline{u}}$, respectively, and $d$ is a function bounded from above by a negative constant (see Section 2.2). As a consequence, the Jacobian (P.25) vanishes where and only where one of $\rho$, $\underline{\rho}$ vanishes. The function $\underline{\rho}$ is given on $\underline{C}_0$ by the initial data and while positive on $\underline{C}_0 \setminus S_{0,0}$, vanishes to 1st order at $S_{0,0}$, the last being a manifestation of the singular nature of the surface $\partial_{-}\mathcal{B}$. The function $\rho$ on $\underline{C}_0$ represents 1st derived data on $\underline{C}_0$ (see Chapter 5) and vanishes there to 0th order. It turns out that these are the only places where $\underline{\rho}$, $\rho$ vanish in $R_{\delta,\delta} \times S^{n-1}$.

A smooth solution of the coupled characteristic and wave systems once obtained then represents a solution of the original Eulerian equations in standard rectangular coordinates which is smooth in $\mathcal{N} \setminus \underline{\mathcal{C}}$ but singular at $\underline{\mathcal{C}}$ with the transversal derivatives of the $\beta_\mu$ being only Hölder continuous of exponent 1/2 at $\underline{\mathcal{C}}$ and, in addition, a stronger singularity at $\partial_{-}\mathcal{B}$, namely the blow up of the derivatives of the $\beta_\mu$ at $\partial_{-}\mathcal{B}$ in the direction tangential to $\underline{\mathcal{C}}$ but transversal to $\partial_{-}\mathcal{B}$. In particular, the new solution is smooth at the shock hypersurface $\mathcal{K}$ except at its past boundary $\partial_{-}\mathcal{B}$, as is the prior solution which holds in the other side of $\mathcal{K}$, the past side, $\mathcal{K} \setminus \partial_{-}\mathcal{B}$, lying in the interior of the domain of the maximal development.

We remark that the roles which the two collinear pairs of null vectorfields $(N, \underline{N})$ and $(L, \underline{L})$ play in the monograph is different. The vectorfields $N$ and $\underline{N}$, being non-degenerate in terms of rectangular coordinates, are used to define a frame field, and their rectangular components are smooth functions of the $(\underline{u}, u, \vartheta)$ coordinates. On the other hand, the vectorfields $L$ and $\underline{L}$, which are smooth when expressed in terms of $(\underline{u}, u, \vartheta)$ coordinates, are used in the role of differential operators.

The wave system consists of the equations

$$dx^\mu \wedge d\beta_\mu = 0, \quad h^{-1}(dx^\mu, d\beta_\mu) = 0 \tag{P.26}$$

expressed in terms of the representation (P.14), that is,

$$\begin{aligned}
(Lx^\mu)\underline{L}\beta_\mu - (\underline{L}x^\mu)L\beta_\mu &= 0, \\
(Lx^\mu)\not{d}\beta_\mu - (\not{d}x^\mu)L\beta_\mu &= 0, \\
(\underline{L}x^\mu)\not{d}\beta_\mu - (\not{d}x^\mu)\underline{L}\beta_\mu &= 0, \\
\not{d}x^\mu \wedge \not{d}\beta_\mu &= 0, \\
\frac{1}{2}\left\{(Lx^\mu)\underline{L}\beta_\mu + (\underline{L}x^\mu)L\beta_\mu\right\} &= a(\not{d}x^\mu, \not{d}\beta_\mu)_{\not{h}},
\end{aligned} \tag{P.27}$$

where

$$\not{h} = h_{\mu\nu}\not{d}x^\mu \otimes \not{d}x^\nu \tag{P.28}$$

is the induced acoustical metric on the surfaces $S_{\underline{u},u}$, a positive-definite metric, and

$$a = -\frac{1}{2}h(L, \underline{L}) = c\rho\underline{\rho}. \tag{P.29}$$

Equations (P.26) are equations for the functions $\beta_\mu : \mu = 0, \ldots, n$ to be satisfied by these together with the functions $x^\mu : \mu = 0, \ldots, n$ on the Lorentzian manifold $(\mathcal{N}, h)$ independently of the choice of a local coordinate system. In $\mathcal{N} \setminus \underline{\mathcal{C}}$ the functions $x^\mu : \mu = 0, \ldots, n$ themselves can be chosen as coordinates, in which case these equations reduce to

$$\frac{\partial\beta_\mu}{\partial x^\nu} - \frac{\partial\beta_\nu}{\partial x^\mu} = 0, \quad h^{\mu\nu}\frac{\partial\beta_\mu}{\partial x^\nu} = 0, \tag{P.30}$$

which are the Eulerian equations in the irrotational case, the 2nd of these equations representing the differential mass conservation law. Here, however, we are imposing the form (P.27) of these equations, that is, using the representation (P.14) we are considering the $\beta_\mu : \mu = 0, \ldots, n$ and the $x^\mu : \mu = 0, \ldots, n$ as functions of $(\underline{u}, u, \vartheta)$. In view of the fact that in equations (P.27) the 1st derivatives of the $x^\mu : \mu = 0, \ldots, n$ as well as the 1st derivatives of the $\beta_\mu : \mu = 0, \ldots, n$ appear, the wave system couples to the characteristic system to principal terms.

On $\underline{C}_0$, $\vartheta$ is defined by the condition that it is constant along the generators of $\underline{C}_0$, together with the condition that the restriction of $\vartheta$ to $S_{0,0} = \partial_-\mathcal{B}$ is a diffeomorphism of $\partial_-\mathcal{B}$ onto $S^{n-1}$. We then have

$$b|_{\underline{C}_0} = 0. \tag{P.31}$$

The initial data for the coupled characteristic and wave systems on $\underline{C}_0 = \underline{\mathcal{C}}$ for the new solution then consist of the pair $((x^\mu : \mu = 0, \ldots, n), (\beta_\mu : \mu = 0, \ldots, n))$ on $\underline{C}_0$ which is that induced by the prior solution.

The characteristic system for the pair $((x^\mu : \mu = 0, \ldots, n),\ b)$, together with the $(h_{\mu\nu} : \mu, \nu = 0, \ldots, n)$ which enter this system through the $(N^\mu : \mu = 0, \ldots, n)$ and the $(\underline{N}^\mu : \mu = 0, \ldots, n)$, manifest a new kind of differential geometric structure which involves the interaction of two geometric structures on the same underlying manifold, the 1st of these structures being the background Galilean structure in the case of the non-relativistic theory, the background Minkowskian structure in the case of the relativistic theory, and the other being the Lorentzian geometry deriving from the acoustical metric. As for the $(\beta_\mu : \mu = 0, \ldots, n)$ of the wave system, this is the set of functions obtained by evaluating the 1-form $\beta$ on the set of translation fields of the background structure. Proposition 2.1 with a translation field substituted for $X$ plays a central role in our approach. The proposition asserts that if $X$ is a vectorfield generating isometries of the background structure then

$$\Box_{\tilde{h}}\beta(X) = 0, \tag{P.32}$$

where $\tilde{h}$ is a metric in the conformal class of the acoustical metric $h$. This wave equation, a differential consequence of the wave system, is the basis for the derivation of the energy estimates on which our entire approach to control the solution is based.

The derivatives of $((x^\mu : \mu = 0, \dots, n),\ b)$ are controlled through the acoustical structure equations, the subject of Chapter 3. These are differential consequences of the characteristic system, bringing out more fully the interaction of the two geometric structures. We have the induced metric $\not{h}$ on the surfaces $S_{\underline{u},u}$ and the functions

$$\lambda = c\underline{\rho}, \quad \underline{\lambda} = c\rho \tag{P.33}$$

(see (P.20)) introduced already in Chapter 2. While $\not{h}$ refers only to the acoustical structure, the functions $\lambda$, $\underline{\lambda}$, involve the interaction of the two geometric structures. These are acoustical quantities of 0th order. The first variation equations (Proposition 3.1) express $\not{\mathcal{L}}_L \not{h}$, $\not{\mathcal{L}}_{\underline{L}} \not{h}$ in terms of $\chi$, $\underline{\chi}$, the two 2nd fundamental forms of $S_{\underline{u},u}$. The torsion forms $\eta$, $\underline{\eta}$ represent the connection in the normal bundle of $S_{\underline{u},u}$ in terms of the vectorfields $L$, $\underline{L}$ which along $S_{\underline{u},u}$ constitute sections of this bundle. The commutator

$$[L, \underline{L}] = \not{\mathcal{L}}_T b, \quad T = L + \underline{L} = \frac{\partial}{\partial \underline{u}} + \frac{\partial}{\partial u} \tag{P.34}$$

(see (P.21)) is expressed in terms of $\eta$, $\underline{\eta}$. While the quantities $\chi$, $\underline{\chi}$, $\eta$, $\underline{\eta}$ refer only to the acoustical structure, the structure equations assume a non-singular form only in terms of the quantities $\tilde{\chi}$, $\underline{\tilde{\chi}}$, $\tilde{\eta}$, $\underline{\tilde{\eta}}$ which involve both structures. The former are related to the latter as follows: up to order 1, remainders depending only on the $d\beta_\mu : \mu = 0, \dots, n$, $\chi$ is equal to $\rho\tilde{\chi}$, $\underline{\chi}$ is equal to $\underline{\rho\tilde{\chi}}$, $\eta$ is equal to $\rho\tilde{\eta}$, $\underline{\eta}$ is equal to $\underline{\rho\tilde{\eta}}$. In fact, the only order-1 quantities appearing in the 1st and 3rd remainders are the $L\beta_\mu : \mu = 0, \dots, n$, and the only order-1 quantities appearing in the 2nd and 4th remainders are the $\underline{L}\beta_\mu : \mu = 0, \dots, n$. Moreover, $\tilde{\eta}$, $\underline{\tilde{\eta}}$, are expressed in terms of $\not{d}\lambda$, $\not{d}\underline{\lambda}$ (see Section 3.2). Thus $\lambda$, $\underline{\lambda}$, $\tilde{\chi}$, $\underline{\tilde{\chi}}$ are the primary acoustical quantities, the 1st two being of 0th order, the 2nd two of 1st order.

Proposition 3.3, the propagation equations for $\lambda$ and $\underline{\lambda}$, plays a central role. This proposition expresses $L\lambda$ and $\underline{L\lambda}$ in terms of 1st-order quantities with vanishing 1st-order acoustical part. Proposition 3.4, the second variation equations, also plays a central role. This proposition likewise expresses $\not{\mathcal{L}}_L \tilde{\chi}$ and $\not{\mathcal{L}}_{\underline{L}} \underline{\tilde{\chi}}$ in terms of 2nd-order quantities with vanishing 2nd-order acoustical part. The acoustical structure equations are completed by the Codazzi and Gauss equations of Propositions 3.6 and 3.7. Denoting by $\not{D}$ the covariant derivative operator of $(S_{\underline{u},u}, \not{h})$, $\not{D}\tilde{\chi}$, $\not{D}\underline{\tilde{\chi}}$ are 3-covariant tensorfields on each $S_{\underline{u},u}$ and the Codazzi equations express the 3-covariant tensorfields obtained from $\not{D}\tilde{\chi}$, $\not{D}\underline{\tilde{\chi}}$ by antisymmetrizing with respect to the 1st two entries in terms of 2nd-order quantities with vanishing 2nd-order acoustical part. The Gauss equation likewise expresses the curvature of $(S_{\underline{u},u}, \not{h})$ in terms of a 2nd-order quantity with vanishing 2nd-order acoustical part. The Codazzi and Gauss equations trivialize in the case of $n = 2$ spatial dimensions.

The boundary conditions on $\mathcal{K}$ are analyzed in Chapter 4. The jumps $\triangle\beta_\mu : \mu = 0, \ldots, n$ in the rectangular components of the 1-form $\beta$ across $\mathcal{K}$ are subject to two conditions, one of which is linear and the other nonlinear. The linear jump condition decomposes into the two conditions

$$\not{d}x^\mu \triangle\beta_\mu = 0, \quad T^\mu \triangle\beta_\mu = 0. \tag{P.35}$$

As a consequence of the 1st of (P.35), $\triangle\beta_\mu$ can be expressed as a linear combination of the components

$$\epsilon = N^\mu \triangle\beta_\mu, \quad \underline{\epsilon} = \underline{N}^\mu \triangle\beta_\mu. \tag{P.36}$$

We denote by $r$ the ratio

$$r = -\frac{\epsilon}{\underline{\epsilon}}. \tag{P.37}$$

In reference to the 2nd of (P.35), $T^\mu = Tx^\mu$ are the rectangular components of the vectorfield $T$ and are given, in view of (P.23), (P.34), by

$$T^\mu = \rho N^\mu + \underline{\rho}\underline{N}^\mu. \tag{P.38}$$

Then in view of (P.33) the 2nd of (P.35) is equivalent to the following boundary condition for $\underline{\lambda}$:

$$r\underline{\lambda} = \lambda \quad : \text{ on } \mathcal{K}. \tag{P.39}$$

We remark here that the propagation equations for $\lambda$ and for $\tilde{\chi}$ are supplemented by initial conditions on $\underline{C}_0$, while the propagation equations for $\underline{\lambda}$ and for $\underline{\tilde{\chi}}$ are supplemented by boundary conditions on $\mathcal{K}$. The boundary condition for $\underline{\tilde{\chi}}$ is first derived in Chapter 10 in the case of 2 spatial dimensions and afterwards, in Chapter 11, in general. The derivation is by applying $\not{D}$ to the 1st of (P.35). The boundary condition for $\underline{\tilde{\chi}}$ takes the form of a relation between $r\underline{\tilde{\chi}}$ and $\tilde{\chi}$ on $\mathcal{K}$ analogous to (P.39) (see (11.34)). The nonlinear jump condition takes the form of a relation between $\epsilon$ and $\underline{\epsilon}$ which, in the setting of the shock development problem, is shown to be equivalent to

$$\epsilon = -j(\underline{\epsilon})\underline{\epsilon}^2, \quad \text{hence } r = j(\underline{\epsilon})\underline{\epsilon}, \tag{P.40}$$

where $j$ is a smooth function (see Proposition 4.2).

The jump $\triangle\beta_\mu$ is a function on $\mathcal{K}$, which at a given point on $\mathcal{K}$ represents the difference of $\beta_\mu$, defined by the new solution which holds in the future of $\mathcal{K}$, at the point, from the corresponding quantity, which we denote by $\beta'_\mu$, for the prior solution which holds in the past of $\mathcal{K}$, at the same point in the background spacetime, namely the same point in Galilei spacetime in the non-relativistic theory, the same point in Minkowski spacetime in the relativistic theory. From [Ch-Mi], [Ch-S], the $\beta'_\mu$ are smooth functions of the coordinates $(t, u', \vartheta')$ where the level sets $C'_{u'}$ are outgoing acoustically null hypersurfaces, and $\vartheta'$, the range of which is $S^{n-1}$, is constant along each generator of each $C'_{u'}$, and the restriction of $\vartheta'$ to each surface $S'_{t,u'} = \Sigma_t \bigcap C'_{u'}$ is a diffeomorphism onto $S^{n-1}$. Here we denote by $\Sigma_t$ the level sets of $t$, the hyperplanes of absolute simultaneity in the non-relativistic theory, parallel spacelike hyperplanes in Minkowski spacetime in the relativistic theory. The spatial rectangular coordinates are likewise smooth functions $x'^i(t, u', \vartheta') : i = 1, \ldots, n$ on the domain of the maximal development up to its boundary. Now according to our construction in Section 2.5, for $c > 0$, $C_c$, the $c$-level set of $u$ in $\mathcal{N}$, is the extension across $\underline{\mathcal{C}} = \underline{C}_0$ of $C'_c$, the $c$-level set of $u'$ in the domain of the prior maximal development. At $\partial_{-}\mathcal{B} = S_{0,0}$, $u'$ vanishes and $\vartheta' = \vartheta$. On the other hand, on $\mathcal{K} \setminus S_{0,0}$, $u' < 0$. Therefore $u'$ and $\vartheta'$ induced on $\mathcal{K}$ by the prior solution are of the form (see (P.16))

$$u' = w(\tau, \vartheta), \quad \vartheta' = \psi(\tau, \vartheta) \tag{P.41}$$

and we have

$$w(0, \vartheta) = 0, \; w(\tau, \vartheta) < 0 \quad : \text{ for } \tau > 0, \quad \psi(0, \vartheta) = \vartheta. \tag{P.42}$$

We denote, in reference to the new solution,

$$f(\tau, \vartheta) = x^0(\tau, \tau, \vartheta), \quad g^i(\tau, \vartheta) = x^i(\tau, \tau, \vartheta) \quad : \; i = 1, \ldots, n. \tag{P.43}$$

Then the point $(\tau, \tau, \vartheta) \in \mathcal{K}$ in terms of the coordinates $(\underline{u}, u, \vartheta)$ represents the same point in the background spacetime as the point $(t, u', \vartheta')$ in terms of the coordinates $(t, u', \vartheta')$, if with (P.41) and $t = f(\tau, \vartheta)$, that is, the temporal rectangular coordinate being the same, we have

$$x'^i(f(\tau, \vartheta), w(\tau, \vartheta), \psi(\tau, \vartheta)) = g^i(\tau, \vartheta) \quad : \; i = 1, \ldots, n, \tag{P.44}$$

that is, also the spatial rectangular coordinates of the two points are the same. We call $w$ and $\psi$ *transformation functions* and equations (P.44) *identification equations*. Since they do not involve derivatives of $(w, \psi)$, they are to determine $(w, \psi)$ pointwise.

Note that our treatment of the shock development problem uses three coordinates systems: the double acoustical $(\underline{u}, u, \vartheta)$ coordinates which cover the domain $\mathcal{N}$ of the new solution, the single acoustical $(t, u', \vartheta')$ covering the domain of the prior solution, and of course the rectangular coordinates which cover the entire background spacetime although the physical quantities are not everywhere smooth functions of them. The $(t, u', \vartheta')$ coordinates enter through the jump conditions, which refer to the prior solution along $\mathcal{K}$. The transformation functions allow us to express quantities along $\mathcal{K}$ which refer to the prior solution in terms of the $(\tau, \vartheta)$ coordinates on $\mathcal{K}$ which correspond to the $(\underline{u}, u, \vartheta)$ coordinates on $\mathcal{N}$.

Thinking of $S^{n-1}$ as the unit sphere in Euclidean $n$-dimensional space, and denoting by $\exp_\vartheta$ the associated exponential map $T_\vartheta S^{n-1} \to S^{n-1}$ we can express the

2nd of (P.41) in the form

$$\vartheta' = \exp_{\vartheta}(\varphi') \tag{P.45}$$

for some $\varphi' \in T_{\vartheta}S^{n-1}$, which is unique provided that the distance on $S^{n-1}$ of $\vartheta'$ from $\vartheta$ is less than $\pi$, as will be the case in our problem. Then setting

$$F^i((\tau,\vartheta),(u',\varphi')) = x'^i(f(\tau,\vartheta),u',\exp_{\vartheta}(\varphi')) - g^i(\tau,\vartheta) \tag{P.46}$$

the identification equations read

$$F^i((\tau,\vartheta),(u',\varphi')) = 0 \quad : \; i = 1,\ldots,n. \tag{P.47}$$

To arrive at a form of these equations to which the implicit function theorem can be directly applied we set

$$u' = \tau v, \quad \varphi' = \tau^3\gamma \tag{P.48}$$

and define $\hat{F}^i((\tau,\vartheta),(v,\gamma))$ by

$$F^i((\tau,\vartheta),(\tau v,\tau^3\gamma)) = \tau^3\hat{F}^i((\tau,\vartheta),(v,\gamma)). \tag{P.49}$$

The identification equations then take their regularized form

$$\hat{F}^i((\tau,\vartheta),(v,\gamma)) = 0 \quad : \; i = 1,\ldots,n, \tag{P.50}$$

given explicitly by Proposition 4.5. The implicit function theorem then applies to determine for each $(\tau,\vartheta) \in \mathcal{K}^{\delta}$ the associated $(v,\gamma)$, $v < 0$, $\gamma \in T_{\vartheta}S^{n-1}$.

By virtue of the equations of the characteristic and wave systems the transversal derivatives on $\underline{C}_0$ of 1st order of $((x^{\mu} : \mu = 0,\ldots,n), b, (\beta_{\mu} : \mu = 0,\ldots,n))$, that is, $((Lx^{\mu} : \mu = 0,\ldots,n), \not{\mathcal{L}}_L b, (L\beta_{\mu} : \mu = 0,\ldots,n))$, are directly expressed in terms of the initial data, with the exception of the pair $(\underline{\lambda}, s_{NL})$, which we call 1st derived data. Here,

$$s_{NL} = N^{\mu}L\beta_{\mu}. \tag{P.51}$$

According to Proposition 5.1, along the generators of $\underline{C}_0$ this pair satisfies a linear homogeneous system of ordinary differential equations. The initial data for this system are at $S_{0,0}$ and vanish as shown in Chapter 4 by appealing to the boundary condition (P.39) for $\underline{\lambda}$. It then follows that the pair $(\underline{\lambda}, s_{NL})$ vanishes everywhere along $\underline{C}_0$. Proceeding to the transversal derivatives on $\underline{C}_0$ of 2nd order, these are all directly expressed in terms of the preceding with the exception of the pair $(T\underline{\lambda}, Ts_{NL})$. Along the generators of $\underline{C}_0$, this pair satisfies a linear system of ordinary differential equations, in principle inhomogeneous but with the same homogeneous part as that in the case of the 1st derived data. Due to the vanishing of the 1st derived data the inhomogeneous terms actually vanish. However, the initial data at $S_{0,0}$ for the pair $(T\underline{\lambda}, Ts_{NL})$ do not vanish, $T\underline{\lambda}\big|_{S_{0,0}}$ being represented by a positive function on

$S^{n-1}$. It then follows that $T\underline{\lambda}$ has a positive minimum on $\underline{C}_0^\delta$ provided that $\delta$ is suitably small depending on the initial data. Here we denote by $\underline{C}_0^\delta$ the part of $\underline{C}_0$ corresponding to $u \leq \delta$, that is, the part contained in $\mathcal{R}_{\delta,\delta}$ (see (P.14)). In general, the transversal to $\underline{C}_0$ derivatives of order $m+1$ are all directly expressed in terms of the preceding with the exception of the pair $(T^m\underline{\lambda}, T^m s_{NL})$. Along the generators of $\underline{C}_0$,this pair satisfies a linear inhomogeneous system of ordinary differential equations with the same homogeneous part as that in the case of the 1st derived data and with inhomogeneous part directly expressed in terms of the preceding. The initial data for this system consist of the pair $(T^m\underline{\lambda}, T^m s_{NL})$ at $S_{0,0}$, determination of which requires determining at the same time $(T^{m-1}v, T^{m-1}\gamma)$ at $S_{0,0}$, which brings in the regularized identification equations as well as the boundary condition (P.39) for $\underline{\lambda}$. The problem is analyzed in Section 5.3. The determination of the derived data of arbitrarily high order is required for the construction of a truncated power series approximation to any desired order.

Chapter 6 introduces a new geometric concept, that of a variation field, which plays a central role in the present work. Let $X, Y$ be arbitrary vectorfields on $\mathcal{N}$. We define the *bi-variational stress* associated to the 1-form $\beta$ and to the pair $X, Y$, to be the $T_1^1$-type tensorfield

$$\dot{T} = \tilde{h}^{-1} \cdot \dot{T}_\flat, \tag{P.52}$$

where $\dot{T}_\flat$ is the symmetric 2-covariant tensorfield

$$\dot{T}_\flat = \frac{1}{2}\left(d\beta(X) \otimes d\beta(Y) + d\beta(Y) \otimes d\beta(X) - (d\beta(X), d\beta(Y))_h h\right), \tag{P.53}$$

which depends only on the conformal class of $h$. Here $\beta(X)$, $\beta(Y)$ are the scalar functions which result when we evaluate the 1-form $\beta$ on the vectorfields $X$, $Y$, and $d\beta(X)$, $d\beta(Y)$ are the 1-forms which are the differentials of these functions. We remark that $\dot{T}$ depends (besides on $\beta$ and $(X, Y)$) only on the Lorentzian manifold $(\mathcal{N}, \tilde{h})$, and not on any background structure. We have the identity

$$\operatorname{div}_{\tilde{h}} \dot{T} = \frac{1}{2}\left(\Box_{\tilde{h}}\beta(X)\right) d\beta(Y) + \frac{1}{2}\left(\Box_{\tilde{h}}\beta(Y)\right) d\beta(X). \tag{P.54}$$

If, moreover, $X, Y$ generate isometries of the background structure, then by (P.32),

$$\operatorname{div}_{\tilde{h}} \dot{T} = 0. \tag{P.55}$$

Setting $X$, $Y$ to be the translation fields of the background structure, that is,of Minkowski or Galilean spacetime according to whether we are in the relativistic or the non-relativistic framework, namely the vectorfields

$$X = \frac{\partial}{\partial x^\mu}, \quad Y = \frac{\partial}{\partial x^\nu},$$

we have

$$\beta(X) = \beta_\mu, \quad \beta(Y) = \beta_\nu$$

and we denote the corresponding bi-variational stress by $\dot{T}_{\mu\nu}$. The identity (P.54) takes in this case the form

$$\mathrm{div}_{\tilde{h}}\,\dot{T}_{\mu\nu} = \tfrac{1}{2}(\Box_{\tilde{h}}\beta_\mu)d\beta_\nu + \tfrac{1}{2}(\Box_{\tilde{h}}\beta_\nu)d\beta_\mu. \tag{P.56}$$

The concept of bi-variational stress was introduced in [Ch-A] in the general context of Lagrangian theories of mappings of a manifold $\mathcal{M}$ into another manifold $\mathcal{N}$, as discussed briefly in Section 6.1. This concept derives from a quadratic form associated to a variation of a solution of the corresponding Euler–Lagrange equations. This quadratic form is the canonical stress associated to the Lagrangian of the corresponding linearized theory. Polarization then gives a symmetric bilinear form associated to a pair of variations: the bi-variational stress. Here, the isometries of the background structure generate variations through solutions. The usefulness of the concept of bi-variational stress in the context of a free boundary problem is in conjunction with the concept of variation fields. A *variation field* is here simply a vectorfield $V$ on $\mathcal{N}$ which along $\mathcal{K}$ is tangential to $\mathcal{K}$. Such a vectorfield defines on $\mathcal{K}$ a differential operator interior to $\mathcal{K}$. The reason why variation fields play an essential role in the context of a free boundary problem is that only differential operators interior to the free boundary can be applied to a jump condition. A variation field can be expanded in terms of the translation fields $\partial/\partial x^\mu : \mu = 0, \ldots, n$:

$$V = V^\mu \frac{\partial}{\partial x^\mu}. \tag{P.57}$$

The coefficients $V^\mu : \mu = 0, \ldots, n$ of the expansion are simply the rectangular components of $V$. To the variation field $V$ we associate the column of 1-forms

$$^{(V)}\theta^\mu = dV^\mu \quad : \ \mu = 0, \ldots, n. \tag{P.58}$$

Note that this depends on the background structure. To a variation field $V$ and to the row of functions $(\beta_\mu : \mu = 0, \ldots, n)$ we associate the 1-form

$$^{(V)}\xi = V^\mu d\beta_\mu. \tag{P.59}$$

To the variation field $V$ is associated the $T^1_1$-type tensorfield

$$^{(V)}S = V^\mu V^\nu \dot{T}_{\mu\nu}. \tag{P.60}$$

In view of (P.52), (P.53), (P.59), we have

$$^{(V)}S = \tilde{h}^{-1} \cdot {}^{(V)}S_\flat, \tag{P.61}$$

where $^{(V)}S_\flat$ is the symmetric 2-covariant tensorfield

$$^{(V)}S_\flat = {}^{(V)}\xi \otimes {}^{(V)}\xi - \tfrac{1}{2}({}^{(V)}\xi, {}^{(V)}\xi)_h h, \tag{P.62}$$

which depends only on the conformal class of $h$. The identity (P.56) together with definition (P.58) implies the identity

$$\operatorname{div}_{\tilde{h}}\ {}^{(V)}S = ({}^{(V)}\xi,\ {}^{(V)}\theta^{\mu})_{\tilde{h}}d\beta_{\mu} - ({}^{(V)}\xi, d\beta_{\mu})_{\tilde{h}}\ {}^{(V)}\theta^{\mu} + ({}^{(V)}\theta^{\mu}, d\beta_{\mu})_{\tilde{h}}\ {}^{(V)}\xi + V^{\mu}(\Box_{\tilde{h}}\beta_{\mu})\ {}^{(V)}\xi. \tag{P.63}$$

A basic requirement on the set of variation fields $V$ is that they span the tangent space to $\mathcal{K}$ at each point. The simplest way to achieve this is to choose one of the variation fields, which we denote by $Y$, to be at each point of $\mathcal{N}$ in the linear span of $N$ and $\underline{N}$ and along $\mathcal{K}$ collinear to $T$, and to choose the other variation fields so that at each point of $\mathcal{N}$ they span the tangent space to the surface $S_{\underline{u},u}$ through that point. We thus set

$$Y = \gamma N + \overline{\gamma}\underline{N}. \tag{P.64}$$

In view of (P.39), the requirement that $Y$ is along $\mathcal{K}$ collinear to $T$ reduces to

$$\overline{\gamma} = r\gamma\ :\ \text{along } \mathcal{K}. \tag{P.65}$$

The optimal choice is to set

$$\gamma = 1 \tag{P.66}$$

in which case (P.65) reduces to

$$\overline{\gamma} = r\ :\ \text{along } \mathcal{K}, \tag{P.67}$$

and to extend $\overline{\gamma}$ to $\mathcal{N}$ by the requirement that it be constant along the integral curves of $L$:

$$L\overline{\gamma} = 0. \tag{P.68}$$

In 2 spatial dimensions there is an obvious choice of a variation field to complement $Y$, namely $E$, the unit tangent field of the curves $S_{\underline{u},u}$ (with counterclockwise orientation). In higher dimensions, we complement $Y$ with the $(E_{(\mu)}\ :\ \mu = 0,\dots,n)$ which are $h$-orthogonal projections to the surfaces $S_{\underline{u},u}$ of the translation fields $(\partial/\partial x^{\mu}\ :\ \mu = 0,\dots,n)$ of the background structure. These are given by (see (8.59))

$$E_{(\mu)} = h_{\mu\nu}(\not{d}x^{\nu})^{\sharp}, \tag{P.69}$$

where, given $\zeta \in T^{*}_{q}S_{\underline{u},u}$ we denote $\zeta^{\sharp} = \not{h}_{q}^{-1}\cdot\zeta \in T_{q}S_{\underline{u},u}$, considering $\not{h}_{q}$ as an isomorphism $\not{h}_{q}\colon T_{q}S_{\underline{u},u} \to T^{*}_{q}S_{\underline{u},u}$. The later part of Section 6.2 contains the analysis of the structure forms of the variation fields. The smoothness of these when expressed in the $(\underline{u}, u, \vartheta)$ coordinates is a consequence of the fact that the rectangular components of $N$, $\underline{N}$ and of $E$ or the $E_{(\mu)}$ are smooth functions of the $(\underline{u}, u, \vartheta)$ coordinates. At the end of Section 6.2 it is shown that by virtue of the wave system, control of the components of ${}^{(V)}\xi$ for all $V$ in the chosen set of variation fields provides control of all the components of the $d\beta_\mu : \mu = 0, \ldots, n$, and this is so pointwise, even though the variation fields only span a codimension-1 subspace of the tangent space to $\mathcal{N}$ at each point.

The fundamental energy identities are discussed in Section 6.3. Given a vectorfield $X$, which we call a *multiplier field*, we consider the vectorfield ${}^{(V)}P$ associated to $X$ and to a given variation field $V$ through ${}^{(V)}S$, defined by

$$ {}^{(V)}P = -\,{}^{(V)}S \cdot X. \tag{P.70} $$

We call ${}^{(V)}P$ the *energy current* associated to $X$ and to $V$. Let us denote by ${}^{(V)}Q$ the divergence of ${}^{(V)}P$ with respect to the conformal acoustical metric $\tilde{h} = \Omega h$:

$$ \mathrm{div}_{\tilde{h}}\, {}^{(V)}P = {}^{(V)}Q. \tag{P.71} $$

We have

$$ {}^{(V)}Q = {}^{(V)}Q_1 + {}^{(V)}Q_2 + {}^{(V)}Q_3, \tag{P.72} $$

where

$$ {}^{(V)}Q_1 = -\tfrac{1}{2}\, {}^{(V)}S^\sharp \cdot {}^{(X)}\tilde{\pi}, \tag{P.73} $$

$$ \begin{aligned} {}^{(V)}Q_2 = & -( {}^{(V)}\xi, {}^{(V)}\theta^\mu)_{\tilde{h}} X\beta_\mu + ( {}^{(V)}\xi, d\beta_\mu)_{\tilde{h}}\, {}^{(V)}\theta^\mu(X) \\ & - ( {}^{(V)}\theta^\mu, d\beta_\mu)_{\tilde{h}}\, {}^{(V)}\xi(X), \end{aligned} \tag{P.74} $$

$$ {}^{(V)}Q_3 = -\,{}^{(V)}\xi(X) V^\mu \Box_{\tilde{h}} \beta_\mu. \tag{P.75} $$

In (P.73), ${}^{(V)}S^\sharp$ is the symmetric 2-contravariant tensorfield corresponding to ${}^{(V)}S$,

$$ {}^{(V)}S^\sharp = {}^{(V)}S \cdot \tilde{h}^{-1}, \tag{P.76} $$

and

$$ {}^{(X)}\tilde{\pi} = \mathcal{L}_X \tilde{h} \tag{P.77} $$

is the *deformation tensor* of $X$, the rate of change of the conformal acoustical metric with respect to the flow generated by $X$. The concept of a multiplier field goes back to the fundamental work of Noether [No] connecting symmetries to conserved quantities.

Integrating (P.71) on a domain in $\mathcal{R}_{\delta,\delta}$ of the form

$$ \mathcal{R}_{\underline{u}_1,u_1} = R_{\underline{u}_1,u_1} \times S^{n-1} = \bigcup_{(\underline{u},u)\in R_{\underline{u}_1,u_1}} S_{\underline{u},u}, \tag{P.78} $$

where, with $(\underline{u}_1, u_1) \in R_{\delta,\delta}$ we denote

$$ R_{\underline{u}_1,u_1} = \left\{ (\underline{u}, u) \quad : \ u \in [\underline{u}, u_1], \underline{u} \in [0, u_1] \right\}, \tag{P.79} $$

we obtain the *fundamental energy identity* corresponding to the variation field $V$ and to the multiplier field X:

$$ {}^{(V)}\mathcal{E}^{\underline{u}_1}(u_1) + {}^{(V)}\underline{\mathcal{E}}^{u_1}(\underline{u}_1) + {}^{(V)}\mathcal{F}^{\underline{u}_1} - {}^{(V)}\underline{\mathcal{E}}^{u_1}(0) = {}^{(V)}\mathcal{G}^{\underline{u}_1,u_1}. \tag{P.80} $$

Here, ${}^{(V)}\mathcal{E}^{\underline{u}_1}(u_1)$ and ${}^{(V)}\underline{\mathcal{E}}^{u_1}(\underline{u}_1)$ are the *energies*,

$$ \begin{aligned} {}^{(V)}\mathcal{E}^{\underline{u}_1}(u_1) &= \int_{C_{u_1}^{\underline{u}_1}} \Omega^{(n-1)/2}\, {}^{(V)}S_\flat(X, L), \\ {}^{(V)}\underline{\mathcal{E}}^{u_1}(\underline{u}_1) &= \int_{\underline{C}_{\underline{u}_1}^{u_1}} \Omega^{(n-1)/2}\, {}^{(V)}S_\flat(X, \underline{L}), \end{aligned} \tag{P.81} $$

and ${}^{(V)}\mathcal{F}^{\underline{u}_1}$ is the *flux*,

$$ {}^{(V)}\mathcal{F}^{\underline{u}_1} = \int_{\mathcal{K}^{\underline{u}_1}} \Omega^{(n-1)/2}\, {}^{(V)}S_\flat(X, M), \tag{P.82} $$

where

$$ M = \underline{L} - L \tag{P.83} $$

is a normal to $\mathcal{K}$ pointing to the interior of $\mathcal{K}$. In (P.81) we denote by $C_{u_1}^{\underline{u}_1}$ the part of $C_{u_1}$ corresponding to $\underline{u} \leq \underline{u}_1$ and by $\underline{C}_{\underline{u}_1}^{u_1}$ the part of $\underline{C}_{\underline{u}_1}$ corresponding to $u \leq u_1$. The right-hand side of (P.80) is the *error integral*:

$$ {}^{(V)}\mathcal{G}^{\underline{u}_1,u_1} = \int_{\mathcal{R}_{\underline{u}_1,u_1}} 2a\Omega^{(n+1)/2}\, {}^{(V)}Q. \tag{P.84} $$

The energies are positive semidefinite if the multiplier field $X$ is acoustically timelike, future directed. Because of the acoustically timelike nature of $\mathcal{K}$ (relative to $\mathcal{N}$), without the restriction imposed by a boundary condition at $\mathcal{K}$ the flux integrand is an indefinite (of index 1) quadratic form in ${}^{(V)}\xi$. In Chapter 7 the conditions on $X$ are investigated which make the flux coercive when ${}^{(V)}\xi$ satisfies on $\mathcal{K}$ the boundary condition to be discussed presently.

Recall that a variation field $V$ is along $\mathcal{K}$ tangential to $\mathcal{K}$ and therefore defines on $\mathcal{K}$ a differential operator interior to $\mathcal{K}$. Then applying $V$ to the nonlinear jump condition yields the boundary condition on $\mathcal{K}$ for ${}^{(V)}\xi$. This appears as Proposition 6.2 in the form

$$ {}^{(V)}\xi_+(A_+) = {}^{(V)}\xi_-(A_-), \tag{P.85} $$

where the subscripts $+$ and $-$ denote the future and past sides of $\mathcal{K}$ respectively and $A_\pm$ are the vectorfields

$$ A_\pm = \frac{\triangle I}{\delta} - K_\pm. \tag{P.86} $$

Here $I$ is the particle current

$$I = nu, \tag{P.87}$$

where $n$ is the rest mass density and $u$ the spacetime fluid velocity (see (1.12) in regard to the relativistic theory, (1.134) in regard to the non-relativistic theory). We define $\zeta$ to be the covectorfield along $\mathcal{K}$ such that at each $p \in \mathcal{K}$ the null space of $\zeta_p$ is $T_p\mathcal{K}$, $\zeta_p(U) > 0$ if the vector $U$ points to the interior of $\mathcal{N}$, $\zeta$ being normalized in the relativistic theory to be of unit magnitude with respect to the Minkowski metric and in the non-relativistic theory by the condition that $\overline{\zeta}$, the restriction of $\zeta$ to the $\Sigma_t$, is of unit magnitude with respect to the Euclidean metric. The nonlinear jump condition can then be stated in the form

$$\zeta \cdot \triangle I = 0, \tag{P.88}$$

that is, the vectorfield $\triangle I$ along $\mathcal{K}$ is tangential to $\mathcal{K}$, while the linear jump conditions (P.35) take the form

$$\triangle \beta = \delta \zeta \tag{P.89}$$

for some function $\delta$ on $\mathcal{K}$. This clarifies the meaning of the 1st term on the right in (P.86). As for the 2nd term, $K_\pm$ is a normal, relative to the acoustical metric, vectorfield along the future and past sides of $\mathcal{K}$ respectively, given by

$$K^\mu_\pm = G_\pm (h^{-1}_\pm)^{\mu\nu} \zeta_\nu, \tag{P.90}$$

where $G$ is in the relativistic theory the thermodynamic function

$$G = \frac{n}{h} \tag{P.91}$$

and reduces to $n$ in the non-relativistic theory. In Section 13.1 we revisit the proof of Proposition 6.2, clarifying things further. Proposition 6.3 plays a central role in the analysis of the coercivity of the flux integrand in Chapter 7. This proposition states that the 1st term on the right in (P.86), a vectorfield along $\mathcal{K}$ tangential to $\mathcal{K}$, is timelike, future directed with respect to the acoustical metric defined by the future solution. The vectorfields $A_\pm$ are singular at $S_{0,0}$. In the following we drop the subscript + in reference to quantities defined on $\mathcal{K}$ by the future solution. To express the boundary condition (P.85) in a form which is regular at $S_{0,0}$ we multiply $A$, $A_-$ by $\kappa G^{-1}$, defining

$$B = \kappa G^{-1} A, \quad B_- = \kappa G^{-1} A_-, \tag{P.92}$$

where $\kappa$ is the positive function defined by

$$\kappa K = \tfrac{1}{2} G M, \tag{P.93}$$

noting that by (P.90), $K$ is collinear and in the same sense as $M$ (see (P.83)). The boundary condition (P.85) is then equivalent to its regularized form:

$$ {}^{(V)}\xi(B) = {}^{(V)}\xi_-(B_-) \tag{P.94}$$

and we have

$$B = B_{\|} + B_{\perp}, \quad B_- = B_{\|} + B_{-\perp}, \tag{P.95}$$

where $B_{\|}$ is tangential to $\mathcal{K}$,

$$B_{\perp} = -\tfrac{1}{2}M \tag{P.96}$$

is an exterior normal to $\mathcal{K}$ relative to the acoustical metric defined by the future solution, and $B_{-\perp}$ is a normal to $\mathcal{K}$ in the same sense relative to the acoustical metric defined by the past solution. The forms of the vectorfields $B_{\|}$, $B_{-\perp}$ in a neighborhood of $S_{0,0}$ in $\mathcal{K}$ are analyzed in the last part of Section 6.4.

Section 7.1 determines the necessary and sufficient conditions on the multiplier field $X$ for the flux integrand (see (P.82)) to be coercive under the boundary condition (P.90). This rests on the fundamental work of Gårding [Ga] who first showed in connection with the initial–boundary value problem for the wave equation $\square_g \phi = 0$ on a Lorentzian manifold $(\mathcal{M}, g)$ with timelike boundary $\mathcal{K}$ that the boundary condition of prescribing $B\phi$ on $\mathcal{K}$, where $B$ is a vectorfield along $\mathcal{K}$, is well posed if $B$ is of the form (P.95) with $B_{\|}$ tangential to $\mathcal{K}$ and timelike, future directed while $B_{\perp}$ is an exterior normal to $\mathcal{K}$, for in this case an appropriate energy inequality can be derived. Here, in Proposition 7.1 we give a simple geometric characterization of the set of multiplier fields making the flux integrand coercive. This is briefly as follows. In the tangent space to $\mathcal{N}$ at a point of $\mathcal{K}$ the acoustically timelike, future-directed vectors constitute the interior of a positive cone (the future sound cone). The vectors with a fixed positive $B_{\|}$ component constitute an acoustically spacelike hyperplane intersecting this interior in an open ball. Then the set of multiplier vectors making the flux integrand at the point in question coercive intersects the spacelike hyperplane in the interior of a certain spheroid. The last is contained in the half ball corresponding to the half space of vectors with positive $B_{\perp}$ component. (See Figures 7.1, 7.2, 7.3.)

We then show that a suitable choice for the multiplier field is

$$X = 3L + \underline{L} \quad : \text{ on } \mathcal{N}. \tag{P.97}$$

This corresponds in the limit $\tau \to 0$ to the center of the spheroid. We confine ourselves to this choice in the remainder of the monograph. Coercivity then means that there is a constant $C'$ such that

$$ {}^{(V)}\mathcal{F}'^{\underline{u}_1} = {}^{(V)}\mathcal{F}^{\underline{u}_1} + 2C' \int_{\mathcal{K}^{\underline{u}_1}} \Omega^{(n-1)/2} ({}^{(V)}b)^2 \tag{P.98}$$

is positive definite (see (9.337)). Here,

$$ {}^{(V)}b = {}^{(V)}\xi_-(B_-) = B_-^{\mu} V\beta_{-\mu} \tag{P.99}$$

(see (13.28)). Note that while this corresponds to the known prior solution, the transformation functions, which are unknown, are also involved, a manifestation of the free boundary nature of the problem. Adding the 2nd term on the right in (P.98) to both sides of the energy identity (P.80), the last takes the form

$$
\begin{aligned}
&{}^{(V)}\mathcal{E}^{\underline{u}_1}(u_1) + {}^{(V)}\underline{\mathcal{E}}^{u_1}(\underline{u}_1) + {}^{(V)}\mathcal{F}'^{\underline{u}_1} \\
&\quad = {}^{(V)}\underline{\mathcal{E}}^{u_1}(0) + {}^{(V)}\mathcal{G}^{\underline{u}_1,u_1} + 2C' \int_{\mathcal{K}^{\underline{u}_1}} \Omega^{(n-1)/2}({}^{(V)}b)^2. \qquad \text{(P.100)}
\end{aligned}
$$

The error integral (P.84) decomposes into

$$
{}^{(V)}\mathcal{G}^{\underline{u}_1,u_1} = {}^{(V)}\mathcal{G}_1^{\underline{u}_1,u_1} + {}^{(V)}\mathcal{G}_2^{\underline{u}_1,u_1} + {}^{(V)}\mathcal{G}_3^{\underline{u}_1,u_1} \qquad \text{(P.101)}
$$

according to the decomposition (P.71) of ${}^{(V)}Q$.

The deformation tensor ${}^{(X)}\tilde{\pi}$ of the multiplier field is analyzed in Section 7.2. The error integral ${}^{(V)}\mathcal{G}_1^{\underline{u}_1,u_1}$ is then estimated. In estimates (7.122), (7.124) singular integrals first appear. But, as we shall see below, this is only the tip of the iceberg. In Section 7.3 the error integral ${}^{(V)}\mathcal{G}_2^{\underline{u}_1,u_1}$ is estimated using the results of Section 6.2 on the structure forms of the variation fields.

The *commutation fields* which are used to control the higher-order analogues of the functions $\beta_\mu : \mu = 0, \ldots, n$ are defined in Section 7.1. Commutation fields were first introduced by Klainerman in his derivation in [Kl] of the decay properties of the solutions of the wave equation in Minkowski spacetime using the fact that this equation is invariant under the Poincaré group, the isometry group of Minkowski spacetime, the commutation fields being the vectorfields generating the group action. The scope of commutation fields was substantially extended in [Ch-Kl] where the problem of the stability of the Minkowski metric in the context of the vacuum Einstein equations of general relativity was solved. In that work, while the metric arising as the development of general asymptotically flat initial data does not possess a non-trivial isometry group, nevertheless a large enough subgroup of the scale extended Poincaré group was found and an action of this subgroup which approximates that of an isometry in the sense that the deformation tensors of the vectorfields generating this action, the commutation fields, are appropriately bounded with decay. In the present monograph, denoting (see (P.16)), for $\sigma \in [0, \delta]$,

$$
\mathcal{K}_\sigma^\delta = \left\{(\tau, \sigma + \tau) \quad : \ \tau \in [0, \delta - \sigma]\right\} \times S^{n-1} \qquad \text{(P.102)}
$$

(note that $\mathcal{K}_0^\delta = \mathcal{K}^\delta$), we require that at each point $q \in \mathcal{N}$, $q \in \mathcal{K}_\sigma^\delta$, the set of commutation fields $C$ spans $T_q\mathcal{K}_\sigma^\delta$. As the 1st of the commutation fields we take the vectorfield $T$. The remaining commutation fields are then required to span the tangent space to the $S_{\underline{u},u}$ at each point. In $n = 2$ spatial dimensions we choose $E$ to complement $T$ as a commutation field. For $n > 2$ we choose the $E_{(\mu)} : \mu = 0, \ldots, n$

(see (P.68)) to complement $T$. Thus $E$ for $n = 2$ and the $E_{(\mu)}$ for $n > 2$ play a dual role, being commutation fields as well as variation fields. However, what characterizes the action of a variation field $V$ is the corresponding structure form ${}^{(V)}\theta$, and what characterizes the action of a commutation field $C$ is the corresponding deformation tensor ${}^{(C)}\tilde{\pi} = \mathcal{L}_C \tilde{h}$.

The commutation fields generate higher-order analogues of the functions $\beta_\mu$ : $\mu = 0, \ldots, n$. At order $m+l$ we have, for $n = 2$,

$$ {}^{(m,l)}\beta_\mu = E^l T^m \beta_\mu, \tag{P.103} $$

and for $n > 2$,

$$ {}^{(m,\nu_1\ldots\nu_l)}\beta_\mu = E_{(\nu_l)} \ldots E_{(\nu_1)} T^m \beta_\mu . \tag{P.104} $$

To these and to the variation field $V$ there correspond higher-order analogues of the 1-form ${}^{(V)}\xi$, namely, for $n = 2$,

$$ {}^{(V;m,l)}\xi = V^\mu d\, {}^{(m,l)}\beta_\mu, \tag{P.105} $$

and for $n > 2$,

$$ {}^{(V;m,\nu_1\ldots\nu_l)}\xi = V^\mu d\, {}^{(m,\nu_1\ldots\nu_l)}\beta_\mu . \tag{P.106} $$

The preceding identities (P.56), (P.63), (P.70)–(P.75), (P.80), (P.100)–(P.101) which refer to $\beta_\mu$ and to ${}^{(V)}\xi$ all hold with these higher-order analogues in the role of $\beta_\mu$ and ${}^{(V)}\xi$ respectively, the boundary condition for ${}^{(V;m,l)}\xi$ (case $n = 2$) being of the form

$$ {}^{(V;m,l)}\xi(B) = {}^{(V;m,l)}b, \tag{P.107} $$

where ${}^{(V;m,l)}b$ is analyzed in Section 13.1. Similarly, for $n > 2$, ${}^{(V;m,\nu_1\ldots\nu_l)}\xi$ is of the form

$$ {}^{(V;m,\nu_1\ldots\nu_l)}\xi(B) = {}^{(V;m,\nu_1\ldots\nu_l)}b. \tag{P.108} $$

By virtue of the wave system and its differential consequences, control of the set of ${}^{(V;m,l)}\xi$ ($n = 2$), ${}^{(V;m,\nu_1\ldots\nu_l)}\xi$ ($n > 2$) for $m+l$ up to a given positive integer $k$ gives us control on all derivatives of the $\beta_\mu$ of order up to $k+1$, and this is so pointwise, despite the fact that the commutation fields only span a codimension-1 subspace of the tangent space to $\mathcal{N}$ at each point.

While for the original $\beta_\mu$ we have $\Box_{\tilde{h}}\beta_\mu = 0$, hence the error term ${}^{(V)}Q_3$ (see (P.75)) vanishes, this is no longer true for the higher-order analogues. Instead we have

$$ \Omega a \Box_{\tilde{h}}\, {}^{(m,l)}\beta_\mu = {}^{(m,l)}\tilde{\rho}_\mu \qquad : \text{case } n = 2, \tag{P.109} $$

$$ \Omega a \Box_{\tilde{h}}\, {}^{(m,\nu_1\ldots\nu_l)}\beta_\mu = {}^{(m,\nu_1\ldots\nu_l)}\tilde{\rho}_\mu \qquad : \text{for } n > 2. \tag{P.110} $$

The ${}^{(m,l)}\tilde{\rho}_\mu$, ${}^{(m,\nu_1\ldots\nu_l)}\tilde{\rho}_\mu$, which we call (rescaled) source functions, obey certain recursion formulas, deduced in Section 8.2, which determine them for all $m$ and $l$.

These depend on up to the $(m + l)$th derivatives with respect to the commutation fields of the deformation tensors of the commutation fields.

To summarize, three kinds of vectorfields are used in our energy estimates: the multiplier field $X$ given by (P.97), the variation fields $V$, and the commutation fields $C$. The multiplier field $X$ enters the error integrals ${}^{(V;m,l)}\mathcal{G}_1^{\underline{u}_1,u_1}$ (case $n = 2$), ${}^{(V;m,\nu_1\dots\nu_l)}\mathcal{G}_1^{\underline{u}_1,u_1}$ (case $n > 2$) through its deformation tensor ${}^{(X)}\tilde{\pi}$, no derivatives of this being involved, the variation fields $V$ enter the error integrals ${}^{(V;m,l)}\mathcal{G}_2^{\underline{u}_1,u_1}$ (case $n = 2$), ${}^{(V;m,\nu_1\dots\nu_l)}\mathcal{G}_2^{\underline{u}_1,u_1}$ (case $n > 2$) through their structure forms ${}^{(V)}\theta^\mu$, no derivatives of these being involved, while the commutation fields enter the error integrals ${}^{(V;m,l)}\mathcal{G}_3^{\underline{u}_1,u_1}$ (case $n = 2$), ${}^{(V;m,\nu_1\dots\nu_l)}\mathcal{G}_3^{\underline{u}_1,u_1}$ (case $n > 2$) through their deformation tensors ${}^{(C)}\tilde{\pi}$ and their up to $(m + l)$th derivatives with respect to the commutation fields themselves.

In the last section of Chapter 8, Section 8.4, the error terms at order $m+l$ are discerned which contain the acoustical quantities of highest order, $m+l+1$. These are contained in the error integral ${}^{(V;m,l)}\mathcal{G}_3^{\underline{u}_1,u_1}$ (case $n = 2$), ${}^{(V;m,\nu_1\dots\nu_l)}\mathcal{G}_3^{\underline{u}_1,u_1}$ (case $n > 2$), which by (P.84) and (P.75) are given by

$$ {}^{(V;m,l)}\mathcal{G}_3^{\underline{u}_1,u_1} = -\int_{\mathcal{R}_{\underline{u}_1,u_1}} 2\Omega^{(n-1)/2}\,{}^{(V;m,l)}\xi(X)V^\mu\,{}^{(m,l)}\tilde{\rho}_\mu \quad : \text{case } n = 2, \tag{P.111} $$

$$ {}^{(V;m,\nu_1\dots\nu_l)}\mathcal{G}_3^{\underline{u}_1,u_1} = -\int_{\mathcal{R}_{\underline{u}_1,u_1}} 2\Omega^{(n-1)/2}\,{}^{(V;m,\nu_1\dots\nu_l)}\xi(X)V^\mu\,{}^{(m,\nu_1\dots\nu_l)}\tilde{\rho}_\mu \quad : \text{for } n > 2. \tag{P.112} $$

In the case $n = 2$ the leading terms in ${}^{(m,l)}\tilde{\rho}$ involving the acoustical quantities of highest order are (see (8.149), (8.150))

$$ \text{for } m = 0\text{:}\ \ \tfrac{1}{2}\rho(\underline{L}\beta_\mu)E^l\tilde{\chi} + \tfrac{1}{2}\underline{\rho}(L\beta_\mu)E^l\underline{\tilde{\chi}}, \tag{P.113} $$

$$ \text{for } m \geq 1\text{:}\ \ \rho(\underline{L}\beta_\mu)E^lT^{m-1}E^2\lambda + \underline{\rho}(L\beta_\mu)E^lT^{m-1}E^2\underline{\lambda}. \tag{P.114} $$

In the case $n > 2$ the leading terms in ${}^{(m,\nu_1\dots\nu_l)}\tilde{\rho}$ involving the acoustical quantities of highest order are (see (8.151), (8.152))

$$ \begin{aligned} \text{for } m = 0\text{:}\ \ & \tfrac{1}{2}\rho(\underline{L}\beta_\mu)E_{(\nu_l)}\dots E_{(\nu_1)}\operatorname{tr}\tilde{\chi} + \tfrac{1}{2}\underline{\rho}(L\beta_\mu)E_{(\nu_l)}\dots E_{(\nu_1)}\operatorname{tr}\underline{\tilde{\chi}} \\ & + \frac{ah_{\nu_1,\kappa}}{2c}(\not{d}\beta_\mu)^\sharp\cdot\left(\underline{N}^\kappa\not{d}(E_{(\nu_l)}\dots E_{(\nu_2)}\operatorname{tr}\tilde{\chi}) + N^\kappa\not{d}(E_{(\nu_l)}\dots E_{(\nu_2)}\operatorname{tr}\underline{\tilde{\chi}})\right), \end{aligned} \tag{P.115} $$

$$ \text{for } m \geq 1\text{:}\ \ \rho(\underline{L}\beta_\mu)E_{(\nu_l)}\dots E_{(\nu_1)}T^{m-1}\not{\Delta}\lambda + \underline{\rho}(L\beta_\mu)E_{(\nu_l)}\dots E_{(\nu_1)}T^{m-1}\not{\Delta}\underline{\lambda}. \tag{P.116} $$

We now come to the main analytic method introduced in this monograph. To motivate the introduction of this method, we need first to discuss the difficulties encountered. After this discussion it will become evident that the new analytic method is a natural way to overcome these difficulties. In Chapters 1–8, which we have just reviewed, we denote by $n$ the spatial dimension, a notation which we have followed in the above review. However, in the remainder of the monograph, Chapters 9–14, which we shall review presently, we designate by $n$ the top order of the ${}^{(m,l)}\beta_\mu$, that is, $m + l = n$, and we denote the spatial dimension by $d$. We shall now follow the latter notation.

The difficulties arise in estimating the contribution of the terms involving the top-order (order-$(n+1)$) acoustical quantities, (P.113), (P.114) in the case $d = 2$, (P.115), (P.116) for $d > 2$, to the error integral ${}^{(V;m,l)}\mathcal{G}_3^{\underline{u}_1,u_1}$, ${}^{(V;m,\nu_1\dots\nu_l)}\mathcal{G}_3^{\underline{u}_1,u_1}$. To understand the origin of these difficulties it is advantageous to go directly to Chapter 11 where the approach applicable to any $d \geq 2$ is laid out, an approach which simplifies in the case $d = 2$ where the detailed estimates are deduced in Chapter 10. In regard to (P.115) we must derive appropriate estimates for

$$\not{d}(E_{(\nu_{l-1})} \dots E_{(\nu_1)} \operatorname{tr} \tilde{\chi}), \quad \not{d}(E_{(\nu_{l-1})} \dots E_{(\nu_1)} \operatorname{tr} \underline{\tilde{\chi}}) \quad : \nu_1, \dots, \nu_{l-1} = 0, \dots, d. \tag{P.117}$$

In regard to (P.116) we must derive appropriate estimates for

$$E_{(\nu_l)} \dots E_{(\nu_1)} T^{m-1} \not{\Delta} \lambda, \quad E_{(\nu_l)} \dots E_{(\nu_1)} T^{m-1} \not{\Delta} \underline{\lambda} \quad : \nu_1, \dots, \nu_l = 0, \dots, d. \tag{P.118}$$

Now, as mentioned above, Proposition 3.4, the second variation equations, express $\not{\mathcal{L}}_L \tilde{\chi}$ and $\not{\mathcal{L}}_{\underline{L}} \underline{\tilde{\chi}}$ in terms of 2nd-order quantities with vanishing 2nd-order acoustical part. These imply expressions for $L \operatorname{tr} \tilde{\chi}$ and $\underline{L} \operatorname{tr} \underline{\tilde{\chi}}$ again in terms of 2nd-order quantities with vanishing 2nd-order acoustical part. To be able then to estimate $\operatorname{tr} \tilde{\chi}$, $\operatorname{tr} \underline{\tilde{\chi}}$ in terms of 1st-order quantities, so that we can estimate (P.117) in terms of quantities of the top order $l + 1 = n + 1$, we must express the principal (2nd-order) part of the expressions for $L \operatorname{tr} \tilde{\chi}$ and $\underline{L} \operatorname{tr} \underline{\tilde{\chi}}$ in the forms $-L\hat{f}$ and $-\underline{L}\hat{\underline{f}}$ respectively, up to lower-order terms, with $\hat{f}$ and $\hat{\underline{f}}$ being quantities of 1st order. That this is possible follows from the fact that the quantities (see (11.14))

$$M = \frac{1}{2}\beta_N^2(a\not{\Delta} H - L(\underline{L}H)), \quad \underline{M} = \frac{1}{2}\beta_{\underline{N}}^2(a\not{\Delta} H - \underline{L}(LH)), \tag{P.119}$$

with $H$ a given smooth function of the $\beta_\mu : \mu = 0, \dots, d$, are actually 1st-order quantities. This fact is a direct consequence of the equation $\Box_{\tilde{h}} \beta_\mu = 0$. The functions $\hat{f}$, $\hat{\underline{f}}$ each contain a singular term with coefficients $\lambda^{-1}$, $\underline{\lambda}^{-1}$ respectively. The functions $f = \lambda \hat{f}$, $\underline{f} = \underline{\lambda}\hat{\underline{f}}$ are then regular, and transferring the corresponding terms to the left-hand side, we obtain propagation equations for the quantities

$$\theta = \lambda \operatorname{tr} \tilde{\chi} + f, \quad \underline{\theta} = \underline{\lambda} \operatorname{tr} \underline{\tilde{\chi}} + \underline{f} \tag{P.120}$$

of the form

$$L\theta = R, \quad \underline{L}\underline{\theta} = \underline{R}, \tag{P.121}$$

where $R$, $\underline{R}$ are again quantities of order 1, their 1st-order acoustical parts being given by Proposition 11.1. The leading terms in $R$, $\underline{R}$ from the point of view of behavior as we approach the singularity at $\underline{u} = 0$ (that is, $\underline{C}_0$) and the stronger singularity at $u = 0$ (that is, $S_{0,0}$) are the terms

$$\begin{aligned} 2(L\lambda)\,\mathrm{tr}\,\tilde{\chi} &= 2\lambda^{-1}(L\lambda)(\theta - f) \ \ : \text{in } R, \\ 2(\underline{L}\underline{\lambda})\,\mathrm{tr}\,\tilde{\underline{\chi}} &= 2\underline{\lambda}^{-1}(\underline{L}\underline{\lambda})(\underline{\theta} - \underline{f}) \ \ : \text{in } \underline{R}. \end{aligned} \tag{P.122}$$

To estimate (P.117) we introduce the quantities, of order $l + 1 = n + 1$,

$$\begin{aligned} {}^{(\nu_1\ldots\nu_{l-1})}\theta_l &= \lambda \not{d}\big(E_{(\nu_{l-1})}\ldots E_{(\nu_1)}\,\mathrm{tr}\,\tilde{\chi}\big) + \not{d}\big(E_{(\nu_{l-1})}\ldots E_{(\nu_1)} f\big), \\ {}^{(\nu_1\ldots\nu_{l-1})}\underline{\theta}_l &= \underline{\lambda} \not{d}\Big(E_{(\nu_{l-1})}\ldots E_{(\nu_1)}\,\mathrm{tr}\,\tilde{\underline{\chi}}\Big) + \not{d}\Big(E_{(\nu_{l-1})}\ldots E_{(\nu_1)} \underline{f}\Big). \end{aligned} \tag{P.123}$$

These quantities then satisfy propagation equations of the form

$$\begin{aligned} \not{\mathcal{L}}_L\, {}^{(\nu_1\ldots\nu_{l-1})}\theta_l &= {}^{(\nu_1\ldots\nu_{l-1})}R_l, \\ \not{\mathcal{L}}_{\underline{L}}\, {}^{(\nu_1\ldots\nu_{l-1})}\underline{\theta}_l &= {}^{(\nu_1\ldots\nu_{l-1})}\underline{R}_l, \end{aligned} \tag{P.124}$$

where ${}^{(\nu_1\ldots\nu_{l-1})}R_l$, ${}^{(\nu_1\ldots\nu_{l-1})}\underline{R}_l$ are likewise also of order $l+1 = n+1$ and their leading terms from the point of view of behavior as we approach the singularities are

$$\begin{aligned} 2(L\lambda)\not{d}(E_{(\nu_{l-1})}\ldots E_{(\nu_1)}\,\mathrm{tr}\,\tilde{\chi}) &= 2\lambda^{-1}(L\lambda)\big({}^{(\nu_1\ldots\nu_{l-1})}\theta_l - \not{d}(E_{(\nu_{l-1})}\ldots E_{(\nu_1)} f)\big), \\ 2(\underline{L}\underline{\lambda})\not{d}(E_{(\nu_{l-1})}\ldots E_{(\nu_1)}\,\mathrm{tr}\,\tilde{\underline{\chi}}) &= 2\underline{\lambda}^{-1}(\underline{L}\underline{\lambda})\big({}^{(\nu_1\ldots\nu_{l-1})}\underline{\theta}_l - \not{d}(E_{(\nu_{l-1})}\ldots E_{(\nu_1)} \underline{f})\big). \end{aligned} \tag{P.125}$$

As mentioned above, Proposition 3.3, the propagation equations for $\lambda$, $\underline{\lambda}$, express $L\lambda$ and $\underline{L}\underline{\lambda}$ in terms of 1st-order quantities with vanishing 1st-order acoustical part. These imply expressions for $L\not{\Delta}\lambda$ and $\underline{L}\not{\Delta}\underline{\lambda}$ in terms of 3rd-order quantities with vanishing 3rd-order acoustical part. To be able then to estimate $\not{\Delta}\lambda$, $\not{\Delta}\underline{\lambda}$ in terms of 2nd-order quantities, so that we can estimate (P.118) in terms of quantities of the top order $m + l + 1 = n + 1$, we must express the principal (3rd-order) part of the expressions for $L\not{\Delta}\lambda$ and $\underline{L}\not{\Delta}\underline{\lambda}$ in the form $L\hat{j}$ and $\underline{L}\hat{\underline{j}}$ respectively, up to lower-order terms, with $\hat{j}$ and $\hat{\underline{j}}$ being quantities of 2nd order. That this is possible follows again from the fact that the quantities $M$ and $\underline{M}$ defined by (P.119) are actually 1st-order quantities. The functions $\hat{j}$, $\hat{\underline{j}}$ each contain a singular term with coefficient $\lambda^{-1}$, $\underline{\lambda}^{-1}$ respectively. The functions $j = \lambda\hat{j}$, $\underline{j} = \underline{\lambda}\hat{\underline{j}}$ are then regular, and transferring the corresponding terms to the left-hand side, we obtain propagation equations for the quantities

$$\nu = \lambda\not{\Delta}\lambda - j, \quad \underline{\nu} = \underline{\lambda}\not{\Delta}\underline{\lambda} - \underline{j} \tag{P.126}$$

of the form

$$L(\nu - \tau) = I, \quad \underline{L}(\underline{\nu} - \underline{\tau}) = \underline{I}, \tag{P.127}$$

where $\tau$, $\underline{\tau}$ are quantities of order 1, while $I$, $\underline{I}$ are quantities of order 2, their 2nd-order acoustical parts being given by Proposition 11.2. The leading terms in $I$, $\underline{I}$ from the point of view of behavior as we approach the singularity at $\underline{u} = 0$ (that is $\underline{C}_0$) and the stronger singularity at $u = 0$ (that is $S_{0,0}$) are the terms

$$2(L\lambda)\not{\Delta}\lambda = 2\lambda^{-1}(L\lambda)(\nu + j) \ : \text{in } I, \qquad 2(\underline{L}\underline{\lambda})\not{\Delta}\underline{\lambda} = 2\underline{\lambda}^{-1}(\underline{L}\underline{\lambda})(\underline{\nu} + \underline{j}) \ : \text{in } \underline{I}. \quad \text{(P.128)}$$

To estimate (P.118) we introduce the quantities, of order $m + l + 1 = n + 1$,

$$\begin{aligned} {}^{(\nu_1\ldots\nu_l)}\nu_{m-1,l+1} &= \lambda E_{(\nu_l)}\ldots E_{(\nu_1)}T^{m-1}\not{\Delta}\lambda - E_{(\nu_l)}\ldots E_{(\nu_1)}T^{m-1}j, \\ {}^{(\nu_1\ldots\nu_l)}\underline{\nu}_{m-1,l+1}* &= \underline{\lambda}E_{(\nu_l)}\ldots E_{(\nu_1)}T^{m-1}\not{\Delta}\underline{\lambda} - E_{(\nu_l)}\ldots E_{(\nu_1)}T^{m-1}\underline{j}. \end{aligned} \quad \text{(P.129)}$$

These quantities then satisfy propagation equations of the form

$$\begin{aligned} L\left( {}^{(\nu_1\ldots\nu_l)}\nu_{m-1,l+1} - {}^{(\nu_1\ldots\nu_l)}\tau_{m-1,l+1}\right) &= {}^{(\nu_1\ldots\nu_l)}I_{m-1,l+1}, \\ \underline{L}\left( {}^{(\nu_1\ldots\nu_l)}\underline{\nu}_{m-1,l+1} - {}^{(\nu_1\ldots\nu_l)}\underline{\tau}_{m-1,l+1}\right) &= {}^{(\nu_1\ldots\nu_l)}\underline{I}_{m-1,l+1}, \end{aligned} \quad \text{(P.130)}$$

where ${}^{(\nu_1\ldots\nu_l)}\tau_{m-1,l+1}$, ${}^{(\nu_1\ldots\nu_l)}\underline{\tau}_{m-1,l+1}$ are quantities of order $m + l = n$, while ${}^{(\nu_1\ldots\nu_l)}I_{m-1,l+1}$, ${}^{(\nu_1\ldots\nu_l)}\underline{I}_{m-1,l+1}$ are quantities of order $m + l + 1 = n + 1$ and their leading terms from the point of view of behavior as we approach the singularities are

$$\begin{aligned} &2(L\lambda)\not{d}(E_{(\nu_l)}\ldots E_{(\nu_1)}T^{m-1}\not{\Delta}\lambda) \\ &\quad = 2\lambda^{-1}(L\lambda)\left( {}^{(\nu_1\ldots\nu_l)}\nu_{m-1,l+1} + E_{(\nu_l)}\ldots E_{(\nu_1)}T^{m-1}j\right), \\ &2(\underline{L}\underline{\lambda})\not{d}(E_{(\nu_l)}\ldots E_{(\nu_1)}T^{m-1}\not{\Delta}\underline{\lambda}) \\ &\quad = 2\underline{\lambda}^{-1}(\underline{L}\underline{\lambda})\left( {}^{(\nu_1\ldots\nu_l)}\underline{\nu}_{m-1,l+1} + E_{(\nu_l)}\ldots E_{(\nu_1)}T^{m-1}\underline{j}\right). \end{aligned} \quad \text{(P.131)}$$

Now, in accordance with the discussion following (P.25), we have

$$\underline{\lambda} \sim \underline{u}, \quad \lambda \sim u^2, \quad \text{(P.132)}$$

where we denote by $\sim$ the ratio being bounded above and below by positive constants. The propagation equations for $\lambda$, $\underline{\lambda}$ of Proposition 3.3 then imply

$$L\lambda = O(\underline{u}), \quad \underline{L}\underline{\lambda} = O(\underline{u}). \quad \text{(P.133)}$$

Then, integrating the 1st of (P.124) from $\underline{C}_0$, the contribution of the term

$$-2\lambda^{-1}(L\lambda)\not{d}(E_{(\nu_{l-1})}\ldots E_{(\nu_1)}f)$$

from the 1st of (P.125) to $\| {}^{(\nu_1\ldots\nu_{l-1})}\theta_l\|_{L^2(S_{\underline{u}_1,u})}$ is bounded by

$$Cu^{-2}\int_0^{\underline{u}_1} \|\not{d}(E_{(\nu_{l-1})}\ldots E_{(\nu_1)}f)\|_{L^2(S_{\underline{u},u})}\underline{u}\, d\underline{u}. \quad \text{(P.134)}$$

Now from the expression for the function $f$ of Proposition 11.1 we see that the term

$$-\frac{1}{2}\beta_N^2 \underline{L}H$$

in $f$ makes the leading contribution, and this contribution to (P.134) is bounded, up to lower-order terms, by

$$Cu^{-2}\int_0^{\underline{u}_1}\sum_{\nu_l}\|\underline{N}^\mu \underline{L}\ {}^{(0,\nu_1\dots\nu_{l-1}\nu_l)}\beta_\mu\|_{L^2(S_{\underline{u},u})}\underline{u}\,d\underline{u} \tag{P.135}$$

(compare with (10.371)). From the 2nd of (P.81) and from (P.62) with ${}^{(0,\nu_1\dots\nu_l)}\beta_\mu$, ${}^{(V;0,\nu_1\dots\nu_l)}\xi$ in the roles of $\beta_\mu$, ${}^{(V)}\xi$, in view of (P.97), we have

$${}^{(V;0,\nu_1\dots\nu_l)}\underline{\mathcal{E}}^{u_1}(\underline{u}) = \int_{\underline{C}_{\underline{u}}^{u_1}}\Omega^{(d-1)/2}\left(({}^{(V;0,\nu_1\dots\nu_l)}\xi_{\underline{L}})^2 + 3a|{}^{(V;0,\nu_1\dots\nu_l)}\not{\xi}|^2\right) \tag{P.136}$$

(${}^{(V)}\not{\xi}$ is the 1-form ${}^{(V)}\xi$ induced on the $S_{\underline{u},u}$). Now, the quantity

$$\|\underline{N}^\mu \underline{L}\ {}^{(0,\nu_1\dots\nu_l)}\beta_\mu\|_{L^2(S_{\underline{u},u})} \tag{P.137}$$

in the integrand in (P.135) can only be estimated through

$$\|\ {}^{(Y;0,\nu_1\dots\nu_l)}\xi_{\underline{L}}\|_{L^2(S_{\underline{u},u})}. \tag{P.138}$$

Comparing with (P.64), in view of the fact that by (P.66)–(P.68) and (P.132) we have

$$\overline{\gamma}\sim u, \tag{P.139}$$

we conclude that (P.137) can only be bounded in terms of

$$Cu^{-1}\|\ {}^{(Y;0,\nu_1\dots\nu_l)}\xi_{\underline{L}}\|_{L^2(S_{\underline{u},u})}; \tag{P.140}$$

therefore (P.135) can only be bounded in terms of

$$Cu^{-3}\sum_{\nu_l}\int_0^{\underline{u}_1}\|\ {}^{(Y;0,\nu_1\dots\nu_{l-1}\nu_l)}\xi_{\underline{L}}\|_{L^2(S_{\underline{u},u})}\underline{u}\,d\underline{u}. \tag{P.141}$$

This bounds the leading contribution to $\|\ {}^{(\nu_1\dots\nu_{l-1})}\theta_l\|_{L^2(S_{\underline{u}_1,u})}$. The corresponding contribution to $\|\ {}^{(\nu_1\dots\nu_{l-1})}\theta_l\|_{L^2(\underline{C}_{\underline{u}_1}^{u_1})}$ is then bounded by

$$C\sum_{\nu_l}\left\{\int_{\underline{u}_1}^{u_1}\left(u^{-3}\int_0^{\underline{u}_1}\|\ {}^{(Y;0,\nu_1\dots\nu_{l-1}\nu_l)}\xi_{\underline{L}}\|_{L^2(S_{\underline{u},u})}\underline{u}\,d\underline{u}\right)^2du\right\}^{1/2}. \tag{P.142}$$

To bound the integral in the square root we must use the Schwarz inequality to replace it by (1/3 times)

$$\begin{aligned}
&\underline{u}_1^3 \int_{\underline{u}_1}^{u_1} u^{-6} \left( \int_0^{\underline{u}_1} \| \,^{(Y;0,\nu_1\ldots\nu_{l-1}\nu_l)}\xi_{\underline{L}} \|^2_{L^2(S_{\underline{u},u})} d\underline{u} \right) du \\
&= \underline{u}_1^3 \int_0^{\underline{u}_1} \left( \int_{\underline{u}_1}^{u_1} u^{-6} \| \,^{(Y;0,\nu_1\ldots\nu_{l-1}\nu_l)}\xi_{\underline{L}} \|^2_{L^2(S_{\underline{u},u})} du \right) d\underline{u} \\
&\leq \underline{u}_1^{-3} \int_0^{\underline{u}_1} \| \,^{(Y;0,\nu_1\ldots\nu_{l-1}\nu_l)}\xi_{\underline{L}} \|^2_{L^2(C_{\underline{u}}^{u_1})} d\underline{u} \\
&\leq C \underline{u}_1^{-3} \int_0^{\underline{u}_1} \,^{(Y;0,\nu_1\ldots\nu_{l-1}\nu_l)}\underline{\mathcal{E}}^{u_1}(\underline{u}) d\underline{u}. \qquad \text{(P.143)}
\end{aligned}$$

Then (P.142) is bounded by

$$C \sum_{\nu_l} \underline{u}_1^{-3/2} \left\{ \int_0^{\underline{u}_1} \,^{(Y;0,\nu_1\ldots\nu_{l-1}\nu_l)}\underline{\mathcal{E}}^{u_1}(\underline{u}) d\underline{u} \right\}^{1/2}. \qquad \text{(P.144)}$$

In regard to (P.112) with $m = 0$, $V = Y$, the factor $Y^\mu \,^{(0,\nu_1\ldots\nu_l)}\tilde{\rho}_\mu$ in the integrand contains the term

$$\tfrac{1}{2}\rho(Y^\mu \underline{L}\beta_\mu) E_{(\nu_l)} \ldots E_{(\nu_1)} \operatorname{tr} \tilde{\underline{\chi}} \qquad \text{(P.145)}$$

contributed by the 1st term in (P.115). Here, by (P.64), (P.66),

$$Y^\mu \underline{L}\beta_\mu = N^\mu \underline{L}\beta_\mu + \overline{\gamma}\underline{N}^\mu \underline{L}\beta_\mu = \lambda \operatorname{tr} \not{s} + \overline{\gamma} s_{\underline{N}\underline{L}}, \qquad \text{(P.146)}$$

where

$$\not{s} = \not{d}x^\mu \otimes \not{d}\beta_\mu, \quad \text{hence} \quad \operatorname{tr} \not{s} = (\not{d}x^\mu, \not{d}\beta_\mu)_{\not{h}}, \qquad \text{(P.147)}$$

and (see (P.51))

$$s_{\underline{N}\underline{L}} = \underline{N}^\mu \underline{L}\beta_\mu. \qquad \text{(P.148)}$$

Expression (P.145) contributes through (P.123) the term

$$\tfrac{1}{2} s_{\underline{N}\underline{L}} \rho \overline{\gamma} \lambda^{-1} E_{(\nu_l)} \cdot \,^{(\nu_1\ldots\nu_{l-1})}\theta_l. \qquad \text{(P.149)}$$

In view of (P.132), (P.139), this term contributes to $^{(Y;0,\nu_1\ldots\nu_l)}\mathcal{G}_3^{\underline{u}_1,u_1}$ (see (P.112)) through the $\underline{L}$ component of $X$ (see (P.97)) a term which can only be bounded in terms of

$$\begin{aligned}
&C \int_{\mathcal{R}_{\underline{u}_1,u_1}} \underline{u} u^{-1} | \,^{(Y;0,\nu_1,\ldots,\nu_l)}\xi_{\underline{L}} | \, | E_{\nu_l} \cdot \,^{(\nu_1\ldots\nu_{l-1})}\theta_l | \\
&\leq C \int_0^{\underline{u}_1} \| \,^{(Y;0,\nu_1\ldots\nu_l)}\xi_{\underline{L}} \|_{L^2(C_{\underline{u}}^{u_1})} \| E_{(\nu_l)} \cdot \,^{(\nu_1\ldots\nu_{l-1})}\theta_l \|_{L^2(C_{\underline{u}}^{u_1})} d\underline{u}. \qquad \text{(P.150)}
\end{aligned}$$

Substituting for $\| {}^{(\nu_1\ldots\nu_{l-1})}\theta_l \|_{L^2(\underline{C}_{\underline{u}}^{u_1})}$ the leading contribution (P.144) we see that (P.150) can in turn only be bounded in terms of

$$C\sum_\mu \int_0^{\underline{u}_1} \left( {}^{(Y;0,\nu_1\ldots\nu_l)}\underline{\mathcal{E}}^{u_1}(\underline{u}) \right)^{1/2} \left\{ \frac{1}{\underline{u}} \int_0^{\underline{u}} {}^{(Y;0,\nu_1\ldots\nu_{l-1}\mu)}\underline{\mathcal{E}}^{u_1}(\underline{u}')d\underline{u}' \right\}^{1/2} \frac{d\underline{u}}{\underline{u}}, \quad \text{(P.151)}$$

a singular integral.

However, this is the lesser of the difficulties one faces in regard to (P.112) with $m = 0$. The greater difficulty arises in estimating the contribution through the factor $V^\mu\ {}^{(0,\nu_1\ldots\nu_l)}\tilde{\rho}_\mu$ in the integrand of the terms

$$\frac{1}{2}\underline{\rho}(V^\mu L\beta_\mu)E_{(\nu_l)}\ldots E_{(\nu_1)}\,\mathrm{tr}\,\underline{\tilde{\chi}} + \frac{a}{2c}h_{\nu_1,\kappa}N^\kappa V^\mu(\not{d}\beta_\mu)^\sharp \cdot \not{d}(E_{(\nu_l)}\ldots E_{(\nu_2)}\,\mathrm{tr}\,\underline{\tilde{\chi}}) \quad \text{(P.152)}$$

contributed by the terms involving $\mathrm{tr}\,\underline{\tilde{\chi}}$ in (P.115). This is because the 2nd of (P.124) must be integrated from $\mathcal{K}$ where a boundary condition holds which equates $r\not{d}(E_{(\nu_{l-1})}\ldots E_{(\nu_1)}\,\mathrm{tr}\,\underline{\tilde{\chi}})$ to $\not{d}(E_{(\nu_{l-1})}\ldots E_{(\nu_1)}\,\mathrm{tr}\,\tilde{\chi})$ to leading terms (see paragraph following (P.39)). In view of definitions (P.123) and (P.39) this takes the form of a relation between $r^2\ {}^{(\nu_1\ldots\nu_{l-1})}\underline{\theta}_l$ and ${}^{(\nu_1\ldots\nu_{l-1})}\theta_l$ on $\mathcal{K}$. Integrating the 2nd of (P.124) from $\mathcal{K}$, the contribution of the term

$$-2\underline{\lambda}^{-1}(\underline{L}\lambda)\not{d}(E_{(\nu_{l-1})}\ldots E_{(\nu_1)}\underline{f})$$

from the 2nd of (P.125) to $\| {}^{(\nu_1\ldots\nu_{l-1})}\underline{\theta}_l \|_{L^2(S_{\underline{u},u_1})}$ is, in view of (P.132), (P.139), bounded by

$$C\int_{\underline{u}}^{u_1} \|\not{d}(E_{(\nu_{l-1})}\ldots E_{(\nu_1)}\underline{f})\|_{L^2(S_{\underline{u},u})}du \quad \text{(P.153)}$$

(compare with (P.134)). This causes no difficulty. The difficulty arises from the boundary term $\| {}^{(\nu_1\ldots\nu_{l-1})}\underline{\theta}_l \|_{L^2(S_{\underline{u},\underline{u}})}$. Taking account of the fact that $r \sim \tau$ along $\mathcal{K}$, we multiply the inequality for $\| {}^{(\nu_1\ldots\nu_{l-1})}\underline{\theta}_l \|_{L^2(S_{\underline{u},u_1})}$ by $\underline{u}^2$ and take the $L^2$ norm with respect to $\underline{u}$ on $[0,\underline{u}_1]$ to seek a bound for $\|\underline{u}^2\ {}^{(\nu_1\ldots\nu_{l-1})}\underline{\theta}_l \|_{L^2(C_{u_1}^{\underline{u}_1})}$. The contribution of the boundary term to this is bounded by a constant multiple of $\|r^2\ {}^{(\nu_1\ldots\nu_{l-1})}\underline{\theta}_l \|_{L^2(\mathcal{K}^{\underline{u}_1})}$, which by the above is bounded in terms of

$$\| {}^{(\nu_1\ldots\nu_{l-1})}\theta_l \|_{L^2(\mathcal{K}^{\underline{u}_1})}. \quad \text{(P.154)}$$

To estimate this we must revisit the argument leading from (P.141) to (P.144). Here we are to set $\underline{u}_1 = u$ in (P.141) to obtain the leading contribution to $\| {}^{(\nu_1\ldots\nu_{l-1})}\theta_l \|_{L^2(S_{u,u})}$. The corresponding contribution to $\| {}^{(\nu_1\ldots\nu_{l-1})}\theta_l \|_{L^2(\mathcal{K}^{\underline{u}_1})}$ is then bounded by

$$C\sum_{\nu_l} \left\{ \int_0^{\underline{u}_1} \left( u^{-3} \int_0^u \| {}^{(Y;0,\nu_1\ldots\nu_{l-1}\nu_l)}\xi_{\underline{L}} \|_{L^2(S_{\underline{u},u})}\underline{u}\,d\underline{u} \right)^2 du \right\}^{1/2}. \quad \text{(P.155)}$$

The integral in the square root is now bounded by (1/3 times)

$$\begin{aligned}
&\int_0^{\underline{u}_1} u^{-3}\left(\int_0^{u} \| {}^{(Y;0,\nu_1\dots\nu_{l-1}\nu_l)}\xi_{\underline{L}}\|^2_{L^2(S_{\underline{u},u})} d\underline{u}\right) du \\
&\quad = \int_0^{\underline{u}_1}\left(\int_{\underline{u}}^{\underline{u}_1} u^{-3}\| {}^{(Y;0,\nu_1\dots\nu_{l-1}\nu_l)}\xi_{\underline{L}}\|^2_{L^2(S_{\underline{u},u})} du\right) d\underline{u} \\
&\quad \leq \int_0^{\underline{u}_1} \underline{u}^{-3}\| {}^{(Y;0,\nu_1\dots\nu_{l-1}\nu_l)}\xi_{\underline{L}}\|^2_{L^2(\underline{C}^{u_1}_{\underline{u}})} d\underline{u} \\
&\quad \leq C\int_0^{\underline{u}_1} \underline{u}^{-3}\, {}^{(Y;0,\nu_1\dots\nu_{l-1}\nu_l)}\underline{\mathcal{E}}^{u_1}(\underline{u}) d\underline{u}
\end{aligned} \tag{P.156}$$

and (P.155) is bounded by

$$C\sum_{\nu_l}\left\{\int_0^{\underline{u}_1} {}^{(Y;0,\nu_1\dots\nu_{l-1}\nu_l)}\underline{\mathcal{E}}^{u_1}(\underline{u})\frac{d\underline{u}}{\underline{u}^3}\right\}^{1/2}. \tag{P.157}$$

This is not only singular but, placing a lower cutoff $\varepsilon > 0$ on $\underline{u}$, blows up like $\varepsilon^{-1}$ as $\varepsilon \to 0$. Moreover, this only bounds $\|\underline{u}^2\, {}^{(\nu_1\dots\nu_{l-1})}\underline{\theta}_l\|_{L^2(C^{\underline{u}_1}_{u_1})}$ rather than $\| {}^{(\nu_1\dots\nu_{l-1})}\underline{\theta}_l\|_{L^2(C^{\underline{u}_1}_{u_1})}$. Nevertheless, from the point of view of scaling, the singular bound (P.157) is similar to the bound (P.144). Also, in view of (P.132), (P.133), the contribution to (P.152) of ${}^{(\nu_1\dots\nu_{l-1})}\underline{\theta}_l$ is pointwise bounded by

$$Cu^2| {}^{(\nu_1\dots\nu_{l-1})}\underline{\theta}_l| \tag{P.158}$$

as compared with (P.149) which is pointwise bounded by

$$C\underline{u}u^{-1}| {}^{(\nu_1\dots\nu_{l-1})}\theta_l|. \tag{P.159}$$

As a consequence, the two contributions to ${}^{(Y;0,\nu_1\dots\nu_l)}\mathcal{G}_3^{\underline{u}_1,u_1}$ are similar from the point of view of scaling, both being borderline.

Similar results are obtained in regard to the error integral ${}^{(V;m,\nu_1\dots\nu_l)}\mathcal{G}_3^{\underline{u}_1,u_1}$ for $m \geq 1$, with $E_{(\nu_l)}\dots E_{(\nu_1)}T^{m-1}\not{\Delta}\lambda$, $E_{(\nu_l)}\dots E_{(\nu_1)}T^{m-1}\not{\Delta}\underline{\lambda}$ playing the roles of $\not{d}(E_{(\nu_{l-1})}\dots E_{(\nu_1)}\operatorname{tr}\tilde{\chi})$, $\not{d}(E_{(\nu_{l-1})}\dots E_{(\nu_1)}\operatorname{tr}\tilde{\underline{\chi}})$, respectively. (Compare (P.116) with (P.115).) In fact, integrating the 1st of (P.130) from $\underline{C}_0$, the contribution of the term

$$2\lambda^{-1}(L\lambda)E_{(\nu_l)}\dots E_{(\nu_1}T^{m-1}j$$

from the 1st of (P.131) to $\| {}^{(\nu_1\dots\nu_l)}\nu_{m-1,l+1}\|_{L^2(S_{\underline{u}_1,u})}$ is bounded by

$$Cu^{-2}\int_0^{\underline{u}_1}\|E_{(\nu_l)}\dots E_{(\nu_1)}T^{m-1}j\|_{L^2(S_{\underline{u},u})}\underline{u}\, d\underline{u}. \tag{P.160}$$

From the expression for the function $j$ of Proposition 11.2 we see that the term

$$\tfrac{1}{4}\beta_N^2 \underline{L}^2 H$$

makes the leading contribution. More precisely, writing $\underline{L}^2 H = \underline{L}TH - \underline{L}LH$ the leading contribution is that of

$$\tfrac{1}{4}\beta_N^2 \underline{L}TH$$

and this contribution to (P.160) is bounded, up to lower-order terms, by

$$Cu^{-2}\int_0^{\underline{u}_1} \|\underline{N}^\mu \underline{L}\ {}^{(m,\nu_1\ldots\nu_l)}\beta_\mu\|_{L^2(S_{\underline{u},u})}\underline{u}\, d\underline{u}. \tag{P.161}$$

Using the fact that

$${}^{(V;m,\nu_1\ldots\nu_l)}\underline{\mathcal{E}}^{u_1}(\underline{u}) = \int_{\underline{C}_{\underline{u}}^{u_1}} \Omega^{(d-1)/2}\left(\left({}^{(V;m,\nu_1\ldots\nu_l)}\xi_{\underline{L}}\right)^2 + 3a\left|{}^{(V;m,\nu_1\ldots\nu_l)}\not{\xi}\right|^2\right), \tag{P.162}$$

we deduce, in analogy with (P.141), that we can bound the leading contribution to $\|\,{}^{(\nu_1\ldots\nu_l)}\nu_{m-1,l+1}\|_{L^2(S_{\underline{u}_1,u})}$ by

$$Cu^{-3}\int_0^{\underline{u}_1} \|\,{}^{(Y;m,\nu_1\ldots\nu_l)}\xi_{\underline{L}}\|_{L^2(S_{\underline{u},u})}\underline{u}\, d\underline{u}. \tag{P.163}$$

By an argument analogous to that leading from (P.141) to (P.144) we then deduce that the leading contribution to $\|\,{}^{(\nu_1\ldots\nu_l)}\nu_{m-1,l+1}\|_{L^2(\underline{C}_{\underline{u}_1}^{u_1})}$ is bounded by

$$C\underline{u}_1^{-3/2}\left\{\int_0^{\underline{u}_1} {}^{(Y;m,\nu_1\ldots\nu_l)}\underline{\mathcal{E}}^{u_1}(\underline{u})d\underline{u}\right\}^{1/2}. \tag{P.164}$$

Then in analogy with (P.151) the corresponding contribution to the error integral ${}^{(Y;m,\nu_1\ldots\nu_l)}\mathcal{G}_3^{\underline{u}_1,u_1}$ is bounded in terms of the mildly singular integral

$$C\int_0^{\underline{u}_1}\left({}^{(Y;m,\nu_1\ldots\nu_l)}\underline{\mathcal{E}}^{u_1}(\underline{u})\right)^{1/2}\left\{\frac{1}{\underline{u}}\int_0^{\underline{u}} {}^{(Y;m,\nu_1\ldots\nu_l)}\underline{\mathcal{E}}^{u_1}(\underline{u}')d\underline{u}'\right\}^{1/2}\frac{d\underline{u}}{\underline{u}}. \tag{P.165}$$

On the other hand, the contribution through the factor $V^\mu\ {}^{(m,\nu_1\ldots\nu_l)}\tilde{\rho}_\mu$ in the integrand in (P.112) for $m \geq 1$ of the term

$$\underline{\rho}(V^\mu L\beta_\mu)E_{(\nu_l)}\ldots E_{(\nu_1)}T^{m-1}\not{d}\,\underline{\lambda}, \tag{P.166}$$

contributed by the term involving $\not{d}\,\underline{\lambda}$ in (P.116), can only be bounded in terms of a severely singular integral. This is because the 2nd of (P.130) must be integrated from $\mathcal{K}$ where a boundary condition which equates $rE_{(\nu_l)}\ldots E_{(\nu_1)}T^{m-1}\not{d}\,\underline{\lambda}$ to

$E_{(\nu_l)} \dots E_{(\nu_1)} T^{m-1} \not{\partial} \lambda$ holds, a differential consequence of the boundary condition (P.39). In view of definitions (P.129) and (P.39) this takes the form of a relation between $r^2\ {}^{(\nu_1 \dots \nu_l)}\underline{\nu}_{m-1,l+1}$ and ${}^{(\nu_1 \dots \nu_l)}\nu_{m-1,l+1}$ on $\mathcal{K}$. The contribution of the boundary term to $\|\underline{u}^2\ {}^{(\nu_1 \dots \nu_l)}\underline{\nu}_{m-1,l+1}\|_{L^2(C^{\underline{u}_1}_{u_1})}$ is then bounded by a constant multiple of $\|r^2\ {}^{(\nu_1 \dots \nu_l)}\underline{\nu}_{m-1,l+1}\|_{L^2(\mathcal{K}^{\underline{u}_1})}$, which in view of this relation is bounded in terms of

$$\|\ {}^{(\nu_1 \dots \nu_l)}\nu_{m-1,l+1}\|_{L^2(\mathcal{K}^{\underline{u}_1})}. \tag{P.167}$$

Proceeding as in the argument leading to (P.157) we conclude that the leading contribution to (P.167) is only bounded in terms of the severely singular integral

$$C \left\{ \int_0^{\underline{u}_1} {}^{(Y;m,\nu_1 \dots \nu_{l-1}\nu_l)}\underline{\mathcal{E}}^{u_1}(\underline{u}) . \frac{d\underline{u}}{\underline{u}^3} \right\}^{1/2}. \tag{P.168}$$

Additional difficulties of the same kind as those discussed above arise indirectly in estimating the contribution of the terms involving the top-order (order-$(n+1)$) derivatives of the transformation functions to the integral which appears as the last term on the right in (P.100) when ${}^{(V)}b$ is replaced by ${}^{(V;m,l)}b$, ${}^{(V;m,\nu_1 \dots \nu_l)}b$ according to (P.107), (P.108).

The new analytic method is designed to overcome the difficulties due to the appearance of the singular integrals. The starting point is the observation that, in view of the fact that these integrals are borderline from the point of view of scaling, they would become regular borderline integrals if the energies ${}^{(V;m,\nu_1 \dots \nu_l)}\mathcal{E}^{\underline{u}}(u)$, ${}^{(V;m,\nu_1 \dots \nu_l)}\underline{\mathcal{E}}^{u}(\underline{u})$ had a growth from the singularities at $\underline{u} = 0$ ($\underline{C}_0$) and $u = 0$ ($S_{0,0}$) like $\underline{u}^{2a_m} u^{2b_m}$ for some sufficiently large exponents $a_m$ and $b_m$, and, moreover, the flux ${}^{(V;m,\nu_1 \dots \nu_l)}\mathcal{F}'^{\tau}$ had a growth from the singularity at $\tau = 0$ ($S_{0,0}$) on $\mathcal{K}$ like $\tau^{2c_m}$, where $c_m = a_m + b_m$. That is, if we assume the stated growth properties then these integrals would give contributions on the right-hand side of the energy identities which would be bounded proportionally to $\underline{u}^{2a_m} u^{2b_m}$, in consistency with the assumption, and moreover with a coefficient which is small if $a_m$ and $b_m$ are accordingly large. This observation seems at first sight irrelevant since on the right-hand side of the energy identities we also have the initial data terms ${}^{(V;m,\nu_1 \dots \nu_l)}\underline{\mathcal{E}}^{u}(0)$ which are independent of $\underline{u}$ (and do not have the assumed growth with $u$). In fact, the growth assumptions for ${}^{(V;m,\nu_1 \dots \nu_l)}\mathcal{E}^{\underline{u}}(u)$, ${}^{(V;m,\nu_1 \dots \nu_l)}\underline{\mathcal{E}}^{u}(\underline{u})$, and ${}^{(V;m,\nu_1 \dots \nu_l)}\mathcal{F}'^{\tau}$ cannot possibly hold.

However, given that the initial data of the problem are expressed as smooth functions of $(u, \vartheta)$, as we have seen above in our review of Chapter 5, the derived data, that is, the $T$-derivatives on $\underline{C}_0$ of up to any desired order $N$ of the unknowns $((x^\mu : \mu = 0, \dots, n),\ b,\ (\beta_\mu : \mu = 0, \dots, n))$ in the characteristic and wave systems, are determined as smooth functions of $(u, \vartheta)$. The $T$-derivatives of the transformation functions $(v, \gamma)$ on $S_{0,0}$ of up to order $N-2$ are also determined at the same time as smooth functions of $\vartheta$. Therefore we can define the $N$th approximants $((x^\mu_N : \mu = 0, \ldots, n),\ b_N,\ (\beta_{\mu,N} : \mu = 0, \ldots, n))$ as the corresponding $N$th degree polynomials in $\tau = \underline{u}$ with coefficients which are known smooth functions of $(\sigma = u - \underline{u}, \vartheta)$. Similarly we can define the $N$th approximants $(v_N, \gamma_N)$ as the corresponding $(N-2)$th degree polynomials in $\tau$ with coefficients which are known smooth functions of $\vartheta$. If the $N$ approximants, so defined, are inserted into the equations of the characteristic and wave systems, these equations fail to be satisfied by errors which are known smooth functions of $(\underline{u}, u, \vartheta)$ and whose derivatives up to order $n$ are bounded by a known constant times $\tau^{N-n+k}$, where $k$ is a fixed integer depending on which equation we are considering. Similarly, inserting the $N$ approximants into the boundary conditions and into the identification equations, these fail to be satisfied by errors which are known smooth functions of $(\tau, \vartheta)$ and whose derivatives up to order $n$ are likewise bounded by a known constant times $\tau^{N-n+k}$, where $k$ is a fixed integer depending on which equation we are considering. Moreover, in connection with the equation $\Box_{\tilde{h}}\beta_\mu = 0$ satisfied by an actual solution, the corresponding $N$th approximant quantity $\Box_{\tilde{h}_N}\beta_{\mu,N}$ is a known smooth function of $(\underline{u}, u, \vartheta)$ and whose derivatives up to order $n$ are bounded by a known constant times $\tau^{N-n+k}$, where $k$ is a fixed integer.

As we shall presently outline, by considering the differences from the $N$th approximants and the associated difference energy identities, the corresponding initial data terms vanish and their role as the inhomogeneous terms in these identities is now played by quantities involving the errors committed by the $N$th approximation, quantities which do have the required growth properties if $N$ is chosen suitably large.

The construction of the truncated power series approximation and the analysis of the errors committed at truncation order $N$ are discussed in detail in Chapter 9, in the case $d = 2$, the extension to the case $d > 2$ being straightforward. Note that in this construction we have, in terms of $(\tau = \underline{u}, \sigma = u - \underline{u}, \vartheta)$ coordinates,

$$T_N = \frac{\partial}{\partial \tau} = T, \quad \Omega_N = \frac{\partial}{\partial \vartheta} = \Omega, \tag{P.169}$$

independently of the approximation. On the other hand, $L_N$, $\underline{L}_N$, $E_N$ depend on the approximation, the 1st two through $b_N$, and the last through $\not{h}_N$, where

$$\not{h}_N = h_{\mu\nu,N}(\Omega x^\mu_N)(\Omega x^\nu_N). \tag{P.170}$$

We then define the difference quantities

$$^{(m,l)}\check{\beta}_\mu = E^l T^m \beta_\mu - E^l_N T^m \beta_{\mu,N}, \tag{P.171}$$

$$^{(V;m,l)}\check{\xi} = V^\mu d\ ^{(m,l)}\check{\beta}_\mu. \tag{P.172}$$

We also define $^{(V;m,l)}\check{S}$ as in (P.61), (P.62) with $^{(V;m,l)}\check{\xi}$ in the role of $^{(V)}\xi$, that is,

$$^{(V;m,l)}\check{S} = \tilde{h}^{-1} \cdot\ ^{(V;m,l)}\check{S}_\flat, \tag{P.173}$$

where

$$^{(V;m,l)}\check{S}_\flat = {}^{(V;m,l)}\check{\xi} \otimes {}^{(V;m,l)}\check{\xi} - \frac{1}{2}({}^{(V;m,l)}\check{\xi}, {}^{(V;m,l)}\check{\xi})_h h. \tag{P.174}$$

Defining, moreover, in analogy with (P.70),

$$^{(V;m,l)}\check{P} = -{}^{(V;m,l)}\check{S} \cdot X, \tag{P.175}$$

we have, in analogy with (P.71)–(P.75),

$$\mathrm{div}_{\tilde{h}}\ {}^{(V;m,l)}\check{P} = {}^{(V;m,l)}\check{Q}, \tag{P.176}$$

where

$$^{(V;m,l)}\check{Q} = {}^{(V;m,l)}\check{Q}_1 + {}^{(V;m,l)}\check{Q}_2 + {}^{(V;m,l)}\check{Q}_3 \tag{P.177}$$

with

$$^{(V;m,l)}\check{Q}_1 = -\tfrac{1}{2}\,{}^{(V;m,l)}\check{S}^\sharp \cdot {}^{(X)}\tilde{\pi}, \tag{P.178}$$

$$\begin{aligned} ^{(V;m,l)}\check{Q}_2 = &-({}^{(V;m,l)}\check{\xi}, {}^{(V)}\theta^\mu)_{\tilde{h}} X\, {}^{(m,l)}\check{\beta}_\mu + ({}^{(V;m,l)}\check{\xi}, d\, {}^{(m,l)}\check{\beta}_\mu)_{\tilde{h}}\, {}^{(V)}\theta^\mu(X) \\ &- ({}^{(V)}\theta^\mu, d\, {}^{(m,l)}\check{\beta}_\mu)_{\tilde{h}}\, {}^{(V;m,l)}\check{\xi}(X), \end{aligned} \tag{P.179}$$

$$^{(V;m,l)}\check{Q}_3 = -{}^{(V;m,l)}\check{\xi}(X) V^\mu \Box_{\tilde{h}}\, {}^{(m,l)}\check{\beta}_\mu. \tag{P.180}$$

We then deduce, in analogy with (P.100), the $(m, l)$ *difference energy identity*

$$\begin{aligned} &^{(V;m,l)}\check{\mathcal{E}}^{\underline{u}_1}(u_1) + {}^{(V;m,l)}\check{\underline{\mathcal{E}}}^{u_1}(\underline{u}_1) + {}^{(V;m,l)}\check{\mathcal{F}}'^{\underline{u}_1} \\ &= {}^{(V;m,l)}\check{\underline{\mathcal{E}}}^{u_1}(0) + {}^{(V;m,l)}\check{\mathcal{G}}^{\underline{u}_1, u_1} + 2C' \int_{\mathcal{K}^{\underline{u}_1}} \Omega^{1/2} ({}^{(V;m,l)}\check{b})^2 \end{aligned} \tag{P.181}$$

(here $d = 2$), where (see (P.81))

$$\begin{aligned} ^{(V;m,l)}\check{\mathcal{E}}^{\underline{u}_1}(u_1) &= \int_{C_{u_1}^{\underline{u}_1}} \Omega^{1/2}\, {}^{(V;m,l)}\check{S}_\flat(X, L) \\ &= \int_{C_{u_1}^{\underline{u}_1}} \Omega^{1/2} \left(3({}^{(V;m,l)}\check{\xi}_L)^2 + a({}^{(V;m,l)}\check{\not{\xi}})^2\right), \\ ^{(V;m,l)}\check{\underline{\mathcal{E}}}^{u_1}(\underline{u}_1) &= \int_{\underline{C}_{\underline{u}_1}^{u_1}} \Omega^{1/2}\, {}^{(V;m,l)}\check{S}_\flat(X, \underline{L}) \\ &= \int_{\underline{C}_{\underline{u}_1}^{u_1}} \Omega^{1/2} \left(3a({}^{(V;m,l)}\check{\not{\xi}})^2 + ({}^{(V;m,l)}\check{\xi}_{\underline{L}})^2\right) \end{aligned} \tag{P.182}$$

are the $(m, l)$ *difference energies*. Also, in (P.181),

$$^{(V;m,l)}\check{\mathcal{F}}'^{\underline{u}_1} = {}^{(V;m,l)}\check{\mathcal{F}}^{\underline{u}_1} + 2C' \int_{\mathcal{K}^{\underline{u}_1}} \Omega^{1/2} ({}^{(V;m,l)}\check{b})^2 \tag{P.183}$$

is positive definite (see (9.337)), ${}^{(V;m,l)}\check{\mathcal{F}}^{\underline{u}_1}$ being the $(m,l)$ *difference flux*

$$ {}^{(V;m,l)}\check{\mathcal{F}}^{\underline{u}_1} = \int_{\mathcal{K}^{\underline{u}_1}} \Omega^{1/2}\,{}^{(V;m,l)}\check{S}_{\flat}(X,M), \tag{P.184} $$

and ${}^{(V;m,l)}\check{b}$ representing the boundary values on $\mathcal{K}$ of ${}^{(V;m,l)}\check{\xi}(B)$ (see (13.1)). Finally, in (P.181),

$$ {}^{(V;m,l)}\check{\mathcal{G}}^{\underline{u}_1,u_1} := \int_{\mathcal{R}_{\underline{u}_1,u_1}} 2a\Omega^{3/2}\,{}^{(V;m,l)}\check{Q} \tag{P.185} $$

is the $(m,l)$ *difference error integral*.

Now, from definition (P.171) and the preceding discussion it follows that for any solution of the problem the functions ${}^{(m,l)}\check{\beta}_\mu$ vanish with all their $T$-derivatives up to order $n$ on $\underline{C}_0$ if we choose $N \geq n$. We then have

$$ {}^{(V;m,l)}\underline{\check{\mathcal{E}}}^{u_1}(0) = 0 \quad : \forall\, m = 0,\ldots,n; \tag{P.186} $$

therefore the $(m,l)$ *difference energy identity* simplifies to

$$ \begin{aligned} &{}^{(V;m,l)}\check{\mathcal{E}}^{\underline{u}_1}(u_1) + {}^{(V;m,l)}\underline{\check{\mathcal{E}}}^{u_1}(\underline{u}_1) + {}^{(V;m,l)}\check{\mathcal{F}}'^{\underline{u}_1} \\ &\quad = {}^{(V;m,l)}\check{\mathcal{G}}^{\underline{u}_1,u_1} + 2C'\int_{\mathcal{K}^{\underline{u}_1}} \Omega^{1/2}({}^{(V;m,l)}\check{b})^2. \end{aligned} \tag{P.187} $$

Expecting that the growth of ${}^{(V;m,l)}\check{\mathcal{E}}^{\underline{u}}(u)$, ${}^{(V;m,l)}\underline{\check{\mathcal{E}}}^{u}(\underline{u})$ is like $\underline{u}^{2a_m}u^{2b_m}$ and that the growth of ${}^{(V;m,l)}\check{\mathcal{F}}'^{\tau}$ is like $\tau^{2c_m}$, we define the weighted quantities

$$ {}^{(V;m,l)}\mathcal{B}(\underline{u}_1,u_1) = \sup_{(\underline{u},u)\in R_{\underline{u}_1,u_1}} \underline{u}^{-2a_m}u^{-2b_m}\,{}^{(V;m,l)}\check{\mathcal{E}}^{\underline{u}}(u), \tag{P.188} $$

$$ {}^{(V;m,l)}\underline{\mathcal{B}}(\underline{u}_1,u_1) = \sup_{(\underline{u},u)\in R_{\underline{u}_1,u_1}} \underline{u}^{-2a_m}u^{-2b_m}\,{}^{(V;m,l)}\underline{\check{\mathcal{E}}}^{u}(\underline{u}), \tag{P.189} $$

and

$$ {}^{(V;m,l)}\mathcal{A}(\tau_1) = \sup_{\tau\in[0,\tau_1]} \tau^{-2(a_m+b_m)}\,{}^{(V;m,l)}\check{\mathcal{F}}'^{\tau}, \tag{P.190} $$

the exponents $a_m$, $b_m$ being non-negative real numbers which are non-increasing with $m$. Of course the above definitions do not make sense unless we already know that the quantities ${}^{(V;m,l)}\check{\mathcal{E}}^{\underline{u}}(u)$, ${}^{(V;m,l)}\underline{\check{\mathcal{E}}}^{u}(\underline{u})$, ${}^{(V;m,l)}\check{\mathcal{F}}'^{\tau}$ have the appropriate growth properties. Making this assumption would introduce a vicious circle into the argument, so this is not what we do. What we actually do is presented in Section 14.11. There we regularize the problem by giving initial data not on $\underline{C}_0$ but on $\underline{C}_{\tau_0}$ for $\tau_0 > 0$, but not exceeding a certain fixed positive number which is much smaller than $\delta$. The initial data on $\underline{C}_{\tau_0}$ is modeled after the restriction to $\underline{C}_{\tau_0}$ of the $N$th approximate solution, the difference being bounded by a fixed constant times $\tau_0^{N-1}$. Similarly considering the $m$th derived data on $\underline{C}_{\tau_0}$, $m = 1, \ldots, n+1$ we show that the difference from the corresponding $N$th approximants on $\underline{C}_{\tau_0}$ is bounded by a fixed constant times $\tau_0^{N-1-m}$. The $(m, l)$ difference energy identity now refers to the domain $\mathcal{R}_{\underline{u}_1, u_1, \tau_0}$ in $\mathcal{N}$ which corresponds to the domain

$$R_{\underline{u}_1, u_1, \tau_0} = \left\{(\underline{u}, u) \quad : \; u \in [\underline{u}, u_1], \; \underline{u} \in [\tau_0, \underline{u}_1]\right\} \tag{P.191}$$

in $\mathbb{R}^2$, so the 1st term on the right in (P.181) is replaced by

$$^{(V;m,l)}\check{\underline{\mathcal{E}}}^{u_1}(\tau_0) \leq C\tau_0^{2(N-2-m)}. \tag{P.192}$$

Moreover, in reference to (P.191), if $\underline{u}_1 - \tau_0$ is suitably small then given a solution defined in $\mathcal{R}_{\underline{u}_1, u_1, \tau_0}$ of the problem with initial data on $\underline{C}_{\tau_0}$, the quantities $^{(V;m,l)}\check{\underline{\mathcal{E}}}^{u}(\underline{u})$, $^{(V;m,l)}\check{\mathcal{E}}^{\underline{u}}(u)$, and $^{(V;m,l)}\check{\mathcal{F}}$, for $(\underline{u}, u) \in R_{\underline{u}_1, u_1, \tau_0}$, are likewise bounded by a constant times $\tau_0^{2(N-2-m)}$. Therefore replacing the supremum over $R_{\underline{u}_1, u_1}$ and the supremum over $[0, \tau_1]$, by the supremum over $\mathcal{R}_{\underline{u}_1, u_1, \tau_0}$ and the supremum over $[\tau_0, \tau_1]$, respectively, in definitions (P.188)–(P.190), and taking $N \geq m + 2 + c_m$, everything now makes sense. In fact, we take $N > m + \frac{5}{2} + c_m$, in which case the modifications in the resulting estimates for the quantities (P.189), (P.190) tend to 0 as $\tau_0 \to 0$. The estimates in the preceding chapters, namely Chapters 10, 12, and 13, concluding with the top-order energy estimates in Chapter 13, are thus derived with the foreknowledge that such modifications involving a very small positive $\tau_0$ will be made which will tend to 0 as $\tau_0 \to 0$. We have chosen not to introduce such a small positive $\tau_0$ from the beginning, because that would have made the exposition more complicated, lengthening the monograph unnecessarily.

The argument relies on the derivation of energy estimates of the top order $m+l = n$ only. Since these are derived in Chapter 13 before the regularization of the problem by the introduction of the small positive $\tau_0$, they are to be thought of as *a priori estimates*. In our exposition, at first we ignore all but the principal terms involved. Thus in Chapters 10 and 11, the aim of which is to derive estimates for the top-order acoustical difference quantities corresponding to (P.123), (P.129),

$$\begin{aligned}
^{(v_1\ldots v_{l-1})}\check{\theta}_l &= {}^{(v_1\ldots v_{l-1})}\theta_l - {}^{(v_1\ldots v_{l-1})}\theta_{l,N}, \\
^{(v_1\ldots v_{l-1})}\check{\underline{\theta}}_l &= {}^{(v_1\ldots v_{l-1})}\underline{\theta}_l - {}^{(v_1\ldots v_{l-1})}\underline{\theta}_{l,N}, \\
^{(v_1\ldots v_l)}\check{\nu}_{m-1,l+1} &= {}^{(v_1\ldots v_l)}\nu_{m-1,l+1} - {}^{(v_1\ldots v_l)}\nu_{m-1,l+1,N}, \\
^{(v_1\ldots v_l)}\check{\underline{\nu}}_{m-1,l+1} &= {}^{(v_1\ldots v_l)}\underline{\nu}_{m-1,l+1} - {}^{(v_1\ldots v_l)}\underline{\nu}_{m-1,l+1,N}
\end{aligned} \tag{P.193}$$

for $d > 2$,

$$\begin{aligned}\check{\theta}_l = \theta_l - \theta_{l,N}, \quad \check{\underline{\theta}}_l = \underline{\theta}_l - \underline{\theta}_{l,N}, \\ \check{\nu}_{m-1,l+1} = \nu_{m-1,l+1} - \nu_{m-1,l+1,N}, \quad \check{\underline{\nu}}_{m-1,l+1} = \underline{\nu}_{m-1,l+1} - \underline{\nu}_{m-1,l+1,N}\end{aligned} \tag{P.194}$$

in the case $d = 2$, we ignore all but the top-order terms, namely those of order $n+1$. An exception occurs in regard to the terms in the boundary estimates for $\check{\underline{\nu}}_{m-1,l+1}$ involving the transformation function differences $\check{\check{f}}$, $\check{v}$, $\check{\gamma}$ (see Proposition 10.9), where the terms of orders 2, 1, 3 lower must be kept because they enter the estimates multiplied by correspondingly lower powers of $\tau$. The only essential change when passing from the case $d = 2$ to $d > 2$, is that in higher dimensions we must also derive $L^2(S_{\underline{u},u})$ estimates for $\lambda \not{D}^{(\nu_1 \ldots \nu_{l-1})}\check{\check{\chi}}_{l-1}$, $\underline{\lambda} \not{D}^{(\nu_1 \ldots \nu_{l-1})}\check{\check{\underline{\chi}}}_{l-1}$ in terms of $L^2(S_{\underline{u},u})$ estimates for ${}^{(\nu_1 \ldots \nu_{l-1})}\check{\theta}_l$, ${}^{(\nu_1 \ldots \nu_{l-1})}\check{\underline{\theta}}_l$ respectively, where

$$\begin{aligned}{}^{(\nu_1 \ldots \nu_{l-1})}\check{\check{\chi}}_{l-1} &= \not{\mathcal{L}}_{E_{(\nu_{l-1})}} \cdots \not{\mathcal{L}}_{E_{(\nu_1)}}\tilde{\chi} - \not{\mathcal{L}}_{E_{(\nu_{l-1}),N}} \cdots \not{\mathcal{L}}_{E_{(\nu_1),N}}\tilde{\chi}_N, \\ {}^{(\nu_1 \ldots \nu_{l-1})}\check{\check{\underline{\chi}}}_{l-1} &= \not{\mathcal{L}}_{E_{(\nu_{l-1})}} \cdots \not{\mathcal{L}}_{E_{(\nu_1)}}\tilde{\underline{\chi}} - \not{\mathcal{L}}_{E_{(\nu_{l-1}),N}} \cdots \not{\mathcal{L}}_{E_{(\nu_1),N}}\tilde{\underline{\chi}}_N.\end{aligned} \tag{P.195}$$

The required estimates are derived using elliptic theory in connection with the Codazzi equations on the $S_{\underline{u},u}$. We must also derive $L^2(S_{\underline{u},u})$ estimates for $\lambda \not{D}^{2\ (\nu_1 \ldots \nu_l)}\check{\lambda}_{m-1,l}$, $\underline{\lambda} \not{D}^{2\ (\nu_1 \ldots \nu_l)}\check{\underline{\lambda}}_{m-1,l}$ in terms of $L^2(S_{\underline{u},u})$ estimates for ${}^{(\nu_1 \ldots \nu_l)}\check{\nu}_{m-1,l+1}$, ${}^{(\nu_1 \ldots \nu_l)}\check{\nu}_{m-1,l+1}$ respectively, where

$$\begin{aligned}{}^{(\nu_1 \ldots \nu_l)}\check{\lambda}_{m-1,l} &= E_{(\nu_l)} \ldots E_{(\nu_1)}T^{m-1}\lambda - E_{(\nu_l),N} \ldots E_{(\nu_1),N}T^{m-1}\lambda_N, \\ {}^{(\nu_1 \ldots \nu_l)}\check{\underline{\lambda}}_{m-1,l} &= E_{(\nu_l)} \ldots E_{(\nu_1)}T^{m-1}\underline{\lambda} - E_{(\nu_l),N} \ldots E_{(\nu_1),N}T^{m-1}\underline{\lambda}_N.\end{aligned} \tag{P.196}$$

The required estimates are derived using elliptic theory for the Laplacian on $S_{\underline{u},u}$.

The aim of Chapter 12 is to derive estimates for the derivatives of top order, $n+1$, of the transformation function differences $\check{\check{f}}$, $\check{v}$, $\check{\gamma}$, and at the same time more precise estimates for the next-to-top-order acoustical differences ${}^{(l-1)}\tilde{\chi}$, ${}^{(l-1)}\tilde{\underline{\chi}} : l = n$, and ${}^{(m,l)}\check{\lambda}$, ${}^{(m,l)}\check{\underline{\lambda}} : m + l = n$, in terms of the top-order difference energies and fluxes. Here, all terms of order up to $n-1$ are ignored and of the terms of order $n$ only those involving acoustical quantities of order $n$, namely the quantities being estimated, are considered. The sharpest estimates for the transformation function differences are needed to appropriately bound, in Chapter 13, the integral on $\mathcal{K}^{\underline{u}_1}$ involving ${}^{(V;m,l)}\check{b}$ which constitutes the 2nd term on the right in (P.187). These estimates require, and are coupled to, the sharpest estimates for the next-to-top-order acoustical differences. For this reason Chapter 12 is the longest chapter of the monograph.

In Chapter 13, which contains the derivation of the top-order energy estimates, we again ignore all but the top-order terms.

The estimates of Chapters 10–13 require taking the exponents $a_m$, $b_m$, $c_m$ to be suitably large. This is in accordance with our preceding heuristic discussion. Moreover, the estimates of Chapters 12 and 13, in particular the top-order energy estimates, require that $\delta$ does not exceed a positive constant which is independent of $m$ and $n$.

In the above, only the treatment of the principal terms is shown, and these are estimated using only the fundamental bootstrap assumptions. The full treatment, which includes all the lower-order terms, uses the complete set of *bootstrap assumptions* stated in Section 14.2. In the same section it is shown how all the lower-order terms are treated. We return to the treatment of the lower-order terms in Section 14.9 where it is shown that the lower-order terms are treated in an optimal manner if we choose all the exponents $a_m$ equal and all the exponents $b_m$ equal:

$$a_m = a, \quad b_m = b \quad : \; m = 0, \ldots, n, \tag{P.197}$$

and this optimal choice is taken from that point on. In Section 14.5, $L^2(S_{\underline{u},u})$ estimates for the ${}^{(m,l)}\check{\beta}_\mu : m + l = n$ are deduced. In Section 14.6, $L^2(S_{\underline{u},u})$ estimates for the $(n-1)$th-order acoustical differences are deduced. In Section 14.7, $L^2(S_{\underline{u},u})$ estimates for all $n$th-order derivatives of the $\check{\beta}_\mu$ are deduced. In Section 14.10 pointwise estimates are deduced and the bootstrap assumptions are recovered as strict inequalities. This recovery, however, requires a further smallness condition on $\delta$ which now depends on $n$. The derivation of pointwise estimates from $L^2(S_{\underline{u},u})$ estimates of derivatives intrinsic to the $S_{\underline{u},u}$ requires $k_0$ derivatives, where $k_0$ is the smallest integer greater than $(d-1)/2$, so in the case $d = 2$ we have $k_0 = 1$. Then, first we set

$$n = n_0 := 2k_0 + 3. \tag{P.198}$$

The smallness condition on $\delta$ needed to recover the bootstrap assumptions is then that corresponding to $n = n_0$. Then given any $n > n_0$, the nonlinear argument having closed at order $n_0$, the bootstrap assumptions are no longer needed, therefore no new smallness conditions on $\delta$ are required to proceed inductively to orders $n_0 + 1, \ldots, n$. We remark that $n_0$ can be lowered to

$$n_0 = k_0 + 1 \tag{P.199}$$

if our treatment of the lower-order terms, which uses only $L^\infty(S_{\underline{u},u})$ and $L^2(S_{\underline{u},u})$ estimates, is refined to make full use of the Sobolev inequalities on the $S_{\underline{u},u}$.

In Section 14.11, first we regularize the problem by giving the initial data on $\underline{C}_{\tau_0}$ as discussed above. We then apply a continuity argument to establish the existence of a solution to this regularized problem defined on the whole of $\mathcal{R}_{\delta,\delta,\tau_0}$. This argument relies on the solution by Majda and Thomann [Ma-Th] of the restricted local shock continuation problem. Our continuity argument relies on [Ma-Th] at two points:

first, to construct a solution on $\mathcal{R}_{\tau_0+\varepsilon,\tau_0+\varepsilon,\tau_0}$ for some $\varepsilon > 0$, which could be very small, even much smaller than $\tau_0$, and second, at the end of the continuity argument, starting with a solution defined on a maximal $\mathcal{R}_{\overline{u}_*,\overline{u}_*,\tau_0}$, to extend a solution which has previously been shown to be defined on $\mathcal{R}_{\overline{u}_*,\overline{u}_*+\varepsilon,\tau_0}$ to the domain corresponding to the triangle

$$\left\{(\underline{u}, u) \quad : \ u \in [\underline{u}, \overline{u}_* + \varepsilon_*], \ \underline{u} \in [\overline{u}_*, \overline{u}_* + \varepsilon_*]\right\}$$

for some $\varepsilon_* > 0$, not exceeding $\varepsilon$. Since the union contains $\mathcal{R}_{\overline{u}_*+\varepsilon_*,\overline{u}_*+\varepsilon_*,\tau_0}$ we then arrive at a contradiction to the maximality of $\overline{u}_*$, unless $\overline{u}_* = \delta$. This double use of a local existence theorem is typical of continuity arguments.

After obtaining a solution on $\mathcal{R}_{\delta,\delta,\tau_0}$ satisfying the appropriate estimates, we take $\tau_0$ to be any member of a sequence $(\tau_{0,m} \ : \ m = M, M+1, M+2, \ldots)$ converging to 0, and pass to the limit in a subsequence to obtain the solution to our problem.

We shall now state an abbreviated version of the theorem, the proof of which is the aim of the present monograph. We denote by $\mathbb{B}^d$ the background spacetime, that is, $(d+1)$-dimensional Minkowski, or Galilei, spacetime according to whether we are in the relativistic, or the non-relativistic context.

**Theorem.** *In any spatial dimension $d \geq 2$, given a prior maximal classical solution there is a $\delta > 0$ and a unique solution of the restricted shock development problem, defined on $\mathcal{R}_{\delta,\delta}$ by the triplet $(x^\mu, b, \beta_\mu)$, where $x^\mu : \mu = 0, \ldots, d$ and $\beta_\mu : \mu = 0, \ldots, d$ are smooth functions and $b$ is a smooth mapping of $R_{\delta,\delta}$ into the space of smooth vectorfields on $S^{d-1}$. The smooth mapping $\mathcal{R}_{\delta,\delta} \to \mathbb{B}^d$ by $(\underline{u}, u, \vartheta) \mapsto (x^\mu(\underline{u}, u, \vartheta) : \mu = 0, \ldots, d)$ has negative Jacobian in $\mathcal{R}_{\delta,\delta}$ except on $\underline{C}_0$ where the Jacobian vanishes. The boundaries $\underline{C}_0^\delta$ and $\mathcal{K}^\delta$ are mapped onto smooth hypersurfaces in $\mathbb{B}^d$, an acoustically null hypersurface $\underline{\mathcal{C}}$ and the shock hypersurface respectively, the latter being acoustically timelike except along its past boundary where it is acoustically null and transverse to the latter. The $\beta_\mu$ expressed through the inverse mapping in terms of rectangular coordinates in $\mathbb{B}^d$ are smooth functions on the image of $\mathcal{R}_{\delta,\delta}$ in $\mathbb{B}^d$ except on $\underline{\mathcal{C}}$, the image of $\underline{C}_0$. The new solution $\beta_\mu$ so expressed extends the prior solution $\beta'_\mu$ in a $C^{1,1/2}$ manner across $\underline{\mathcal{C}}$.*

The full statement of the theorem is found at the end of Section 14.1.

It is to be noted from the proof of the theorem that when only finite differentiability of the data is assumed, the solution is shown to exist in a finite and lower differentiability class. The question then arises of whether the solution possesses the additional derivatives. Presumably it does, but these satisfy bounds weaker than those shown to hold for the lower derivatives.

Another open question is the following. If we consider the least upper bound $\overline{\delta}$ of the set of all $\delta > 0$ such that a smooth solution as above exists on $\mathcal{R}_{\delta,\delta}$, what happens at $C_{\overline{\delta}}^{\overline{\delta}}$, the future boundary of $\mathcal{R}_{\overline{\delta},\overline{\delta}}$? Recall that we have not imposed any smallness conditions on the initial data. Presumably, if appropriate smallness conditions are imposed the answer will be that somewhere on $C_{\underline{\delta}}^{\overline{\delta}}$ there is a singularity signaling the formation of another shock.

We conclude this prologue with a few historical remarks to acknowledge the fact that the methods of the present monograph have their roots in the past and that at best we simply develop things a little further in a particular direction. The method of continuity, using a local existence theorem together with a priori bounds, derived on the basis of bootstrap assumptions which are recovered in the course of the argument, originated in the field of ordinary differential equations, in particular the classical problem of the motion of $N$ point masses under their mutual gravitational attraction formulated by Newton in his *Principia* [Ne]. This corresponds to a Hamiltonian system with Hamiltonian function $H = K + V$ where

$$K = \sum_{\alpha=1}^{N} \frac{|p_\alpha|^2}{2M_\alpha}$$

is the kinetic energy, $p_\alpha$, being the momentum of the mass labeled $\alpha$, a vector in Euclidean 3-dimensional space $\mathbb{E}^3$ at the point $x_\alpha \in \mathbb{E}^3$, the position of the mass $\alpha$, and $M_\alpha > 0$ its mass. Also,

$$V = -\frac{1}{2} \sum_{\alpha \neq \beta} \frac{G M_\alpha M_\beta}{r_{\alpha\beta}}$$

is the potential energy, $r_{\alpha\beta}$ being the distance between the masses $\alpha$ and $\beta$ and $G$ being Newton's gravitational constant. The potential energy is defined when the $x_\alpha$ are distinct. If it is regularized by replacing $r_{\alpha\beta}$ by $\sqrt{r_{\alpha\beta}^2 + \varepsilon^2}$, the simplest application of the method of continuity, without bootstrap assumptions, shows that for any initial condition we have a solution defined for all future time. For the actual problem the same method gives a basic theorem due to Painlevé (see [Si-Mo]) stating that for any initial condition either we have a solution defined for all future time, or the maximal existence interval is $[0, t_*)$, in which case, with $\rho = \min_{\alpha \neq \beta} r_{\alpha\beta}$ the minimal mutual distance, we have $\rho(t) \longrightarrow 0$ as $t \longrightarrow t_*$, so in a sense at time $t_*$ we have a collision. To this day it is still not known whether in the $6N$-dimensional space of initial conditions, the subset leading to a collision in the above sense, has zero measure or positive measure. Incidentally, the *Principia* is the work where power series were first introduced.

Another example of the method of continuity in conjunction with a local existence theorem and a priori bounds is from the field of elliptic partial differential equations. This is the non-parametric minimal hypersurface problem: given a domain in a hyperplane in $\mathbb{E}^{n+1}$ and a graph over the boundary, find an extension to a graph over the interior which is of minimal $n$-dimensional area. The graph is that of the height function above the hyperplane. We can assume that the mean value of the height function on the boundary vanishes. To apply the method of continuity we multiply the boundary height function, the data of the problem, by $t \in [0, 1]$. Then for $t = 0$ we have the trivial solution where the graph coincides with the hyperplane domain. For any $t_0 \in [0, 1]$ for which we have a regular solution, the implicit function theorem plays the role of a local existence theorem giving us a solution for $t$ suitably close to $t_0$. The problem then reduces to that of deriving the appropriate a priori bounds. Once these have been established, the method of continuity yields a solution for all $t \in [0, 1]$, in particular for $t = 1$, which is the original problem. The required a priori bounds were established in two steps. The 1st step was the derivation of gradient bounds on the height function and was done by Bernstein in his fundamental works [Be1], [Be2], [Be3]. The 2nd step was the derivation of Hölder estimates for the gradient. This step was done half a century later, independently by De Giorgi [DG] and by Nash [Na]. While De Giorgi's work was in the framework of minimal surface theory, Nash's motivation was actually a very different problem, the study of the evolution of a viscous, heat conducting compressible fluid, which is why he addressed an analogous problem for parabolic equations and deduced the result for elliptic equations as a corollary in the time-independent case. The complete result for the original problem, which is that mean convexity of the boundary of the domain is both necessary and sufficient for the problem to be solvable for all data, was deduced later by Jenkins and Serrin [Je-Se].

A last example of the method of continuity, which uses a local existence theorem and a priori bounds which are derived on the basis of bootstrap assumptions recovered in the course of the argument, is from the field of hyperbolic partial differential equations, therefore closer to the topic of the present monograph. This is the work [Ch-Kl], mentioned above, on the problem of the stability of the Minkowski metric. In this case the local existence theorem had been established in the basic work of Fourès-Bruhat [FB]. We then consider the least upper bound $t_*$ of the set of all positive times $t_1$ for which we have a solution on the spacetime slab corresponding to the time interval $[0, t_1]$ satisfying the bootstrap assumptions. So if $t_*$ is finite, while the solution extends to the spacetime slab corresponding to $[0, t_*]$, the inequalities corresponding to the bootstrap assumptions are saturated on that slab. The a priori bounds are then derived through a construction on the slab in question which proceeds from the future boundary of the slab, which corresponds to the time $t_*$. This is because the derivation of the a priori bounds uses approximate symmetries, which a posteriori are shown to become exact in the limit $t_* \to \infty$. The a priori bounds show that the inequalities corresponding to the bootstrap assumptions are in fact not saturated on the slab in question. Then after another application of the local existence theorem we arrive at a contradiction to the maximality of $t_*$.