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Estimates for differential operators in half-space. Translated from the German by Darya Apushkinskaya. (English)

EMS Tracts in Mathematics 31. Zürich: European Mathematical Society (EMS) (ISBN 978-3-03719-191-0/hbk; 978-3-03719-691-5/ebook). xvi, 246 p. (2019).

Originally, this book was published as [Abschätzungen für Differentialoperatoren im Halbraum. Übers. aus dem Russischen von Werner Plischke und Ehrhard Herbst. In dt. Spr. hrsg. von G. Wildenhain. Lizenzausg. Basel-Boston-Stuttgart: Birkhäuser Verlag (1982; Zbl 0481.47030)]. Here, an English translation is proposed. The monograph presents a detailed study of many inequalities for differential operators with constant coefficients in $\mathbb{R}^n_+ = \{(x,t),\ t \geq 0\}$. The results are due to the authors. The present edition proposes different possibilities for generalizations to (pseudo)differential operators with variable coefficients in other domains. The monograph is divided into 4 chapters.
Chapter 1, entitled "Estimates for matrix operators", provides necessary and sufficient conditions for the validity of the following vector estimate:

$$\|R(D)u\|^2_{B^{1/2}} \leq C(\|P(D)u\|^2 + \langle\langle Q(D)u\rangle\rangle^2). \tag{1}$$

In (1), $u = (u_1, \dots, u_m) \in C_0^\infty(\mathbb{R}^n_+)$, the symbols $R(\xi,\tau) = \{R_j(\xi,\tau)\}$, $P(\xi,\tau) = \{P_{kj}(\xi,\tau)\}$, $Q(\xi,\tau) = \{Q_{\alpha j}(\xi,\tau)\}$ are $1\times m$, $m \times m$ and $N \times m$ matrices, respectively. Their elements are polynomials of $\tau \in \mathbb{R}^1$ having complex measurable locally bounded coefficients in $\mathbb{R}^{n-1}$ which grow no faster then some power of $|\xi|$ for $|\xi| \to \infty$, $\xi \in \mathbb{R}^{n-1}$. Denote by $\widehat{u}(\xi,t)$ the partial Fourier transform of u with respect to x. Then

$$\|u\|^2_{B^{1/2}} = \int_{\mathbb{R}^{n-1}} \int_0^\infty B(\xi)|\widehat{u}(\xi,t)|^2\,dt\,d\xi,$$

$$\langle\langle u\rangle\rangle^2 = \int_{\mathbb{R}^{n-1}} |\widehat{u}(\xi,0)|^2\,d\xi,\ \ \langle\langle u\rangle\rangle^2_{B^{1/2}} = \int_{\mathbb{R}^{n-1}} B(\xi)|\widehat{u}(\xi,0)|^2\,d\xi.$$

The measurable function $B(\xi) > 0$ a.e.
Section 1.4 contains several examples of operators arising in analysis and mechanics for which the estimates proved in Chapter 1 hold. It concerns some generalized-homogeneous elliptic systems, the Lamé system of static elasticity theory, the Cauchy-Riemann system, the stationary linearized Navier-Stokes system, hyperbolic systems, and others.
Chapter 2 is entitled "Boundary estimates for differential operators". Here, necessary and sufficient conditions for the validity of the estimate

$$\langle\langle R(D)u\rangle\rangle^2_{B^{1/2}} \leq C\left(\sum_{j=1}^m \|P_j(D)u\|^2 + \sum_{\alpha=1}^N \langle\langle Q_\alpha(D)u\rangle\rangle^2\right),\ u \in C_0^\infty(\mathbb{R}^n_+), \tag{2}$$

are proved and an exact description of the set of traces $R(D)u|_{t=0}$ for u belonging to the completion of $C_0^\infty(\mathbb{R}^n_+)$ in the metric $\sum_{j=1}^m \|P_j(D)u\|^2$ is given. $R(\xi,\tau)$, $P_j(\xi,\tau)$, $Q_\alpha(\xi,\tau)$ are scalar polynomials of τ with complex measurable locally bounded in $\mathbb{R}^{n-1}$ coefficients with growth no faster than some power of $|\xi|$ for $|\xi| \to \infty$; $\xi \in \mathbb{R}^{n-1}$. A criterion for the validity of (2) is established in Section 2.2. Consider the case when $m \geq 1$, $Q_\alpha \equiv 0$.

According to Corollary 2.2.8, (2) holds if and only if the following conditions are satisfied:
(1) $R(\xi,\tau) \equiv 0 \pmod{\Pi_+(\xi,\tau)})$, where Π_+ is some polynomial in τ depending on $P_j(\xi,\tau)$, $1 \le j \le m$,
(2) $\sup_\xi B(\xi)\Lambda(\xi) < \infty$, where $\Lambda(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}^1} \frac{\sum_1^m (T_j(\xi,\tau)|^2}{\sum_1^m |P_j(\xi,\tau)|^2}\, d\tau$, $T_j(\xi,\tau)$ are polynomials in τ, explicitly defined in Lemma 2.2.1.
The above-mentioned "trace space" $R(D)u|_{t=0}$ coincides with closure of the linear spaces of functions $\varphi \in C_0^\infty(\mathbb{R}^{n-1})$ that satisfy the inequality:

$$\langle\langle\varphi\rangle\rangle^2_{\Lambda^{-1/2}} = \int_{\mathbb{R}^{n-1}} \frac{|\varphi^1(\xi)|^2}{\Lambda(\xi)}\, d\xi < \infty \text{ w.r.t. the norm } \langle\langle\cdot\rangle\rangle_{\Lambda^{-1/2}}.$$

Chapter 3 is entitled "Dominance of differential operators". Necessary and sufficient conditions for the validity of

$$\|R(D)u\|^2_{B^{1/2}} \le C\left(\sum_{j=1}^m \|P_j(D)u\|^2 + \sum_{\alpha=1}^N \langle\langle Q_\alpha(D)u\rangle\rangle^2\right),\ u \in C_0^\infty(\mathbb{R}^n_+), \tag{3}$$

are found, R, P_j, Q_α are the same as in (2).
As a simple corollary of their results, the authors obtain an interesting theorem of Aronszajn. Let $P_j(\xi,\tau)$, $1 \le j \le m$, be homogeneous polynomials in (ξ,τ) of order J. Then the system of operators $P_j(D)$ is coercive iff, for all $\xi \in \mathbb{R}^{n-1}$, the polynomials $P_j(\xi,\tau)$ have no common roots $(\xi,z) \ne 0$.
Chapter 4 deals with special cases of the estimates (2), (3). The goal of the authors is to specify classes of operators P, R ($Q_\alpha \equiv 0$) for which the corresponding necessary and sufficient conditions for validity of (2), (3) with $(\|u\|^2 + \|Pu\|^2)$ in the right-hand side take a simpler and more explicit form. For example, if $\operatorname{ord}_\tau P(\xi,\tau) = J \ge 1$, $P(\xi,\tau)$ is quasielliptic of type $l \ge 1$, and $R(\xi,\tau) = \tau^s$, $s = 0, 1, \dots, J-1$ in (2), then (2) holds true if and only if

$$B(\xi)(1 + \langle\xi\rangle)^{(2s+1)m/J} \le \text{const.}$$

Here, m is an integer and $\langle\cdot\rangle$ is the norm in $\mathbb{R}^{n-1}$ defined by the quasielliptic polynomial $P(\xi,\tau)$.

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