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Boundary behavior of solutions to elliptic equations in general domains. (English)

EMS Tracts in Mathematics 30. Zürich: European Mathematical Society (EMS) (ISBN 978-3-03719-190-3/hbk; 978-3-03719-690-8/ebook). x, 431 p. (2018).

This book is about the Wiener criterion for elliptic partial differential equations of second-order and for polyharmonic equations. Also moduli of continuity are provided in irregular domains in the Euclidean n-dimensional space. The focus is not on smooth domains: the more irregular, the better.
A boundary point is regular for the solution to the Laplace equation if and only if there exists a barrier function at that point. Perron's method provides a unique solution, which may fail to attain the correct boundary values at some points. Regularity of a boundary point $\xi \in \partial\Omega$ means that, given any continuous boundary values $g : \partial\Omega \to \mathbb{R}$, the solution to the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega \end{cases}$$

attains the correct boundary values $\lim_{x\to\xi} u(x) = g(x)$ as x approaches ξ in the domain Ω. This condition with a barrier is rather implicit. In 1924 N. Wiener gave a necessary and sufficient condition in terms of the electrostatic capacity of that part of the ball $\overline{B}_r = \{x \in \mathbb{R}^n \mid |x - \xi| \leq r\}$ which lies in the complement of Ω. The celebrated Wiener criterion for regularity of the boundary point ξ is that the integral

$$\int_0 \frac{\operatorname{cap}(\overline{B}_r \setminus \Omega)}{\operatorname{cap}(\overline{B}_r)} \frac{dr}{r} = \infty$$

diverges. The same Wiener criterion was extended to uniformly elliptic equations

$$\operatorname{div}\langle \mathbf{A}(x), \nabla u\rangle = 0$$

with bounded measurable coefficients by *W. Littman* et al. [Ann. Sc. Norm. Super. Pisa, Sci. Fis. Mat., III. Ser. 17, 43–77 (1963; Zbl 0116.30302)]. For a harmonic function, vanishing on the boundary near ξ, Maz'ya obtained a *modulus of continuity* in terms of the integral

$$\exp\left[-\frac{n-2}{2(n-1)} \int_r^R \frac{\operatorname{cap}(\overline{B}_r \setminus \Omega)}{\rho^{n-1}} \frac{d\rho}{\rho}\right]$$

appearing in Wiener's criterion. An important tool is an inequality with the capacity, the so-called Maz'ya-Sobolev inequality.
The first three chapters are about the second-order elliptic equations $\operatorname{div}\langle \mathbf{A}(x), \nabla u) = f(x)$. In Chapter 4 quasilinear equations are considered, a prototype being induced by the p-Laplace operator $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$. Here V. Maz'ya succeeded in finding the correct counterpart to the Wiener criterion. In 1970 he proved that the condition

$$\int_0 \left[\frac{\rho - \operatorname{cap}(\overline{B}_r \setminus \Omega)}{r^{n-p}}\right]^{1/(p-1)} \frac{dr}{r} = \infty$$

is sufficient for regularity. It came with a modulus of continuity. The necessity of Maz'ya's Wiener criterion was later proved for $p > n-1$ by Lindqvist and Martio and, finally, for all $p > 1$ by Kilpelainen and Maly. (The cases $p \geq n$ are of no interest now.) The proof of the necessity is not included in this book, but a lot of sharp examples are provided.

More than half of the book is devoted to higher-order equations like the biharmonic equation $\Delta\Delta u = f$ and to polyharmonic equations. Chapter 8 is about certain fractional equations $(-\Delta)^{\alpha} u = f$ and is based on work by Eilertsen, a former student of Maz'ya. Even the Lamé system is treated. While the material in the first four chapters is nowadays easily accessible, the results for polyharmonic equations have not earlier been exposed in text books. They are to the greatest part credited to Maz'ya himself. The book is, in fact, a collection of Maz'ya's own research papers. Introductions to the chapters glue the separate parts together, but the notation and main text is often as in the original papers. Thus it is not unified. The style is succinct and explicit. Some of the original work has til now been available only in Russian. Each chapter ends with useful brief historical comments and references. Since the author is a leading expert on the topic, the treaty is likely to be valuable for the researchers.

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