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BOOK REVIEW

A. Kosyak: "Regular, Quasi-regular and Induced Representations of Infinite-dimensional Groups"

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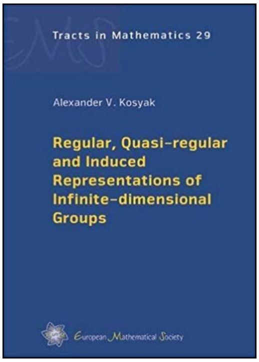

The book under review is a research monograph exploring novel directions and methods in the unitary representation theory of infinite dimensional groups. It develops various aspects of this theory for certain types of groups with a focus on measure theoretic constructions. In particular, a variant of unitary induction is introduced for these groups and irreducibility criteria for representations associated to quasi-invariant measures are obtained.

To see how these results fit into the larger context of unitary representation theory, let us take a look at its historic development. The theory of unitary group representations can be seen as a non-commutative generalization of expansions in Fourier series. It started to develop in the 1920s, largely motivated by its applications to quantum mechanics, where symmetries are modeled in terms of unitary representations. A unitary representation is a homomorphism $U\colon G \to \mathrm{U}(\mathcal{H})$ of a topological group G into the unitary group $\mathrm{U}(\mathcal{H})$ of a complex Hilbert space $\mathcal{H}$ for which all orbit maps $G \to \mathcal{H}$, $g \mapsto U(g)\xi$ are continuous. The typical representation theoretic questions concern decompositions into irreducible representations ($\{0\}$ and $\mathcal{H}$ are the only invariant closed subspaces) and the classification of the irreducible ones. For the circle group $G = \mathbb{T} = \{z \in \mathbb{C}\colon |z| = 1\}$, this is achieved by the abstract Fourier expansion

$$\xi = \sum_{n\in\mathbb{Z}} \xi_n \in \mathcal{H} = \bigoplus_{n\in\mathbb{Z}} \mathcal{H}_n, \quad \mathcal{H}_n = \{\xi \in \mathcal{H}\colon (\forall z \in \mathbb{T})\, U(z)\xi = z^n \xi\}.$$

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The circle is an example of a compact group, and the classical Peter–Weyl Theorem asserts that any unitary representation of a compact group decomposes as a direct sum of irreducible representations, where the eigenspaces have to be replaced by the isotypic subspaces containing all irreducible subrepresentations of the same type. This motivates to consider for a general topological group G the set $\widehat{G}$ of equivalence classes of irreducible unitary representations and to ask similar questions. The situation already becomes more involved for non-compact abelian groups G. In this case $\widehat{G} \cong \mathrm{Hom}(G, \mathbb{T})$ is simply the set of characters, f.i., $\widehat{\mathbb{T}} \cong \mathbb{Z}$ which underlies expansions in Fourier series. For the non-compact group $G = \mathbb{R}$, the character group is $\widehat{G} = \{\chi_x(t) = e^{itx} : x \in \mathbb{R}\} \cong \mathbb{R}$, but the regular representation of $\mathbb{R}$ on $\mathcal{H} = L^2(\mathbb{R})$ by translations $(U_t f)(x) = f(x - t)$ already shows that no irreducible subrepresentation or eigenvectors exist. The decomposition of this representation is achieved by the Fourier transform

$$\mathcal{F}\colon L^2(\mathbb{R}, dx) \to L^2(\mathbb{R}, dp), \qquad \mathcal{F}(f)(p) = \int_{\mathbb{R}} e^{ixp} f(x)\, dx$$

which intertwines translations with multiplications

$$\mathcal{F}(U_t f)(p) = e^{itp} f(p).$$

It can therefore be viewed as a "diagonalization" of the translation operators, where the Dirac distributions $\delta_x(p) = \delta_{x,p}$ (not contained in $L^2(\mathbb{R}, dp)$) play the role of the eigenvectors. These phenomena lead to the concept of a direct integral of Hilbert spaces

$$\mathcal{H} = \int_X^{\oplus} \mathcal{H}_x\, d\mu(x),$$

where $(X, \mathfrak{S}, \mu)$ is a σ-finite measure space. Elements of such spaces are "measurable" functions $X \ni x \mapsto \xi_x \in \mathcal{H}_x$ with $\|\xi\|^2 = \int_X \|\xi(x)\|^2\, d\mu(x) < \infty$. A representation U on such a space is a direct integral $U = \int_X^{\oplus} U_x\, d\mu(x)$ if $(U(g)\xi)_x = U_x(g)\xi_x$ holds for every $g \in G$ and μ-almost every x. In these terms, the Fourier transform establishes an isomorphism $L^2(\mathbb{R}, dp) \cong \int_{\mathbb{R}}^{\oplus} \mathbb{C}\delta_p\, dp$ of representations of $G = \mathbb{R}$.

We are now prepared to formulate the general fundamental questions: For a topological group G, describe the set $\widehat{G}$ of equivalence classes of irreducible unitary representations and find for a given unitary representation a direct integral decomposition $\mathcal{H} = \int_{\widehat{G}}^{\oplus} \mathcal{H}_{[\pi]}\, d\mu([\pi])$. Both problems are quite hard in general. For instance the unitary dual $\widehat{G}$ has very nice accessible descriptions for compact Lie groups (in terms of highest weight representations of semisimple complex Lie algebras) and for nilpotent ones by Kirillov's orbit method (developed in the late 1960s) [2]. Beyond these two classes, the classification results become rather involved, even for the simplest non-compact simple Lie group $\mathrm{SL}_2(\mathbb{R})$ [3]. Presently there is no explicit description of $\widehat{G}$ for all simple connected Lie groups. For an algorithmic classification, we refer to the recent preprint [1].

The decomposition theory of unitary representations largely translates into problems concerning the structure of operator algebras. For a locally compact group G, this technique works very well because there exists a C^*-algebra $C^*(G)$ which has the "same" representations as G. It is obtained by starting from a left invariant measure μ_G on G (Haar measure). This measure specifies an L^1-space endowed with a convolution product

$$(f_1 * f_2)(y) = \int_G f_1(x) f_2(x^{-1}y)\, d\mu_G(x)$$

and a suitable isometric involution $f \mapsto f^*$. A completion process then leads to $C^*(G)$. Now the highly developed representation theory of C^*-algebras can be used to study unitary group representations. In particular, this provides a decomposition as a direct integral of irreducible representations for a suitable Borel structure (= measurable space) on $\widehat{G}$. If $\widehat{G}$ is a standard Borel space (these are the well-behaved ones), G is called a type I group and all algebraic matrix groups have this property. However, there exist non-type I groups, and their representations have to be analyzed with deeper methods from the theory of von Neumann algebras.

Due to its close relations with measure theory, analysis and physics, the unitary representation theory of locally compact groups enjoys diverse applications in a broad range of areas of mathematics and physics. We refer to G. Mackey's nice books [4, 5] for a rich overview. In particular, in quantum mechanics, E. Wigner actually defined elementary particles in terms of irreducible unitary representations of the Poincaré group $P(4) = \mathbb{R}^{1,3} \rtimes \mathrm{SO}_{1,3}(\mathbb{R})^{\uparrow}$ of 4-dimensional Minkowski space [11]. Here the Poincaré group plays the role as a symmetry group of a single particle.

When physicists started to work with Hilbert spaces corresponding to quantum fields and to many particle states, they immediately encountered infinite dimensional symmetry groups such as the symplectic group $\mathrm{Sp}(\mathcal{H}, \omega)$ of an infinite dimensional Hilbert space, groups of smooth maps $C^\infty(\mathbb{S}^1, \mathrm{SU}_2(\mathbb{C}))$ or the diffeomorphism group $\mathrm{Diff}(\mathbb{S}^1)$ of the circle [9, 10]. All these groups are infinite dimensional Lie groups, i.e., manifolds with smooth group operations, hence never locally compact. This is due to the fact that they are modeled on Hausdorff topological vector spaces, and if such a space is locally compact, then it is finite dimensional. The most important drawback of being not locally compact is that these groups do not carry a left invariant Haar measure. Another serious problem is that there is no "structure theory" for infinite dimensional groups such as for finite dimensional Lie groups, whose unitary representation theory can largely be reduced to the classes of solvable and simple Lie groups.

The world of infinite dimensional groups and their representations is simply too diverse, so one cannot expect uniform methods that apply to all groups and all unitary representations. To develop a useful theory, one thus has to focus on specific classes of groups, on specific classes of representations, or even on oth. There are two major approaches:

- the first one is based on Lie theoretic methods with a focus on the representations of the Lie algebra of G. It contains various types of representations that arise naturally in physics [6, 9, 10] and which are specified by positivity conditions on certain operators corresponding to Hamiltonians (positive energy conditions). For such representations, called semibounded in [6], a rather uniform theory can

be developed. In particular smoothing operators [8] can be used to construct C^*-algebras for semibounded representations which behave as nicely as for locally compact groups. We refer to [7] for a recent survey on these methods;
- the second approach focuses on the measure theoretic and the topological environment provided by an infinite dimensional group and produces representations on L^2-spaces. This is what the author does in the present monograph.

A very interesting feature of the measure theoretic side of unitary representation theory is that it changes its flavor when passing from finite to infinite dimensions. For instance finite dimensional homogeneous G-spaces always carry a quasi-invariant measure but in the infinite dimensional context there may be several mutually singular quasi-invariant measures or none at all. Here the finer structures of the groups, such as manifold structures etc., are in the background and the coarser measurable structures are emphasized.

To be more concrete, let G be a topological group and $(X, \mathfrak{S}, \mu)$ be a σ-finite measure space on which G acts by measurable isomorphisms $X \times G \to X$, $(x, g) \mapsto xg$. Further, let μ be a quasi-invariant measure, i.e., the transformed measures μ^g are equivalent to μ. The corresponding quasi-regular unitary representation of G on $L^2(X, \mu)$ is defined by

$$(U_g f)(x) = \Big(\frac{d\mu(xg)}{d\mu(x)}\Big)^{1/2} f(xg). \tag{1}$$

So we can use quasi-invariant measures to construct unitary representations and then analyze their structure, such as irreducibility and direct integral decompositions. To enrich the framework, we can also study an L^2-space $L^2(X, \mu; \mathcal{K})$ of functions with values in a Hilbert space $\mathcal{K}$ and then study representations of the form

$$(U_g f)(x) = \Big(\frac{d\mu(xg)}{d\mu(x)}\Big)^{1/2} J_g(x) f(xg), \tag{2}$$

where $J_g \colon X \to \mathrm{U}(\mathcal{K})$ are measurable functions satisfying a suitable cocycle condition. If the action of G on X is transitive, then $X \cong G/H$ is a homogeneous space, and these representations are called induced representations. Induced representations play a key role in G. Mackey's classification theory of representations of locally compact groups and an important point of the present book is to demonstrate how to extend these methods beyond locally compact groups. Classically, this concerns the case where $X = G/H$ is a homogeneous space, but here one has to work with enlarged spaces $\widetilde{X} = \widetilde{G}/\widetilde{H}$ to have a space supporting a G-quasi-invariant measure (which G/H typically does not). A typical example is the additive group $G = (\ell^2, +)$ which has to be enlarged to $\widetilde{G} = \mathbb{R}^{\mathbb{N}}$ (the full sequence space) to obtain a natural Borel space supporting G-quasi-invariant measures; here Gaussian measures provide examples in abundance.

To single out the interesting situations, one calls the G-action on $(X, \mathfrak{S}, \mu)$ ergodic if the G-invariant elements in $L^\infty(X, \mu)$ are constant. This is necessary for the above construction to produce irreducible representations, but in general far from sufficient.

An interesting new perspective comes in through the group $\mathrm{Aut}(X, \mathfrak{S})^G$ of measurable automorphisms of X commuting with G. One of the author's guiding ideas is that the representation of G on $L^2(X,\mu)$ is expected to be irreducible if μ is G-ergodic and, for every $h \neq \mathrm{id}_X$ in $\mathrm{Aut}(X,\mathfrak{S})^G$, the measures μ^h and μ are mutually singular (Ismagilov's conjecture). Unfortunately this is not always the case (examples are found easily) and a substantial part of the book is concerned with its verification for several classes of groups. This leads to the interesting phenomenon of groups with an irreducible "regular" representation, which never happens for locally compact groups.

The concrete groups studied in this book are

- the groups $B_0^{\mathbb{N}}$ ($B^{\mathbb{N}}$) of $\mathbb{N}\times\mathbb{N}$-matrices of the form $\mathbf{1}+x$, where x is strictly upper triangular with finitely (arbitrarily) many non-zero entries;
- the groups $B_0^{\mathbb{Z}}$ and $B^{\mathbb{Z}}$, where the index set is $\mathbb{Z}$ instead of $\mathbb{N}$;
- the groups $\mathrm{Bor}_0^{\mathbb{N}}$ and $\mathrm{Bor}^{\mathbb{N}}$, where finitely/infinitely many invertible diagonal entries $\neq 1$ are allowed;
- the group $\mathrm{GL}_0(2\infty,\mathbb{R})$ of invertible $\mathbb{Z}\times\mathbb{Z}$-matrices of the form $\mathbf{1}+x$, where x has only finitely many non-zero entries.

Here is a very sketchy overview over the main results of the book:

- The Ismagilov conjecture is verified for the quasi-regular representations of the groups $G = B_0^{\mathbb{N}}$ ($B_0^{\mathbb{Z}}$) on $\widetilde{G} = B^{\mathbb{N}}$ ($B^{\mathbb{Z}}$), where the measures are infinite products of Gaussian measures on the matrix entries (Chap. 2).
- Quasi-regular representations of the groups $G = B_0^{\mathbb{N}}$, $B_0^{\mathbb{Z}}$ and $\mathrm{Bor}_0^{\mathbb{N}}$ are studied on homogeneous spaces $\widetilde{G}/\widetilde{H}$ and criteria for irreducibility are obtained in the spirit of the Ismagilov conjecture (Chaps. 3/4). In Chap. 9 it is shown that the irreducibility results for these representations do not carry over to the analogs of the group $B_0^{\mathbb{N}}$ over a finite field. Similar results are obtained for the action of $\mathrm{GL}_0(2\infty,\mathbb{R})$ on spaces of matrix rows (Chap. 10).
- The regular representation of $G = B_0^{\mathbb{N}}$ on $\widetilde{G} = B^{\mathbb{N}}$ (endowed with a Gaussian product measure) is shown to behave similarly to the regular representation of a locally compact group in the sense that the von Neumann algebras generated by left and right translations are their mutual commutants exchanged by conjugating with an operator J implementing the group inversion (Chap. 5). Under certain assumptions on the measure it is further shown that the von Neumann algebra generated by the regular representation is a hyperfinite factor of type III_1. This is a particularly interesting result because these von Neumann algebras also play a key role in algebraic quantum field theory.
- Induced representations of $B_0^{\mathbb{Z}}$ on "coadjoint orbits" are studied in a similar fashion as in Kirillov's orbit method (Chap. 7).
- The structure of the space $\widehat{G}$ is addressed for the groups $G = B_0^{\mathbb{N}}$ and $B_0^{\mathbb{Z}}$. It is shown that it is not exhausted by the described limit constructions. However, the author introduces a particularly interesting technique of extending representations of G to larger Hilbert–Lie groups $G_2(a)$, so that $\widehat{G}$ is the union of the spaces $\widehat{G_2(a)}$ and it is hoped that the representations of the groups $G_2(a)$ can be classified in a natural way (Chap. 8).

The present book is a reasonably self-contained systematic exposition of results and techniques concerning quasi-regular representations and induced representations of certain infinite dimensional Lie groups. It breaks new ground at the boundaries of our present knowledge about the zoo of representations of these groups. To understand the general principles and the systematics of this theory keeps to be a big challenge.

The author did a remarkable job in writing this book with several very informative survey chapters and in building the results of himself and his coauthors into a natural coherent framework. The book is accessible to advanced graduate students with a good background in measure theory and some basic Lie group theory. It provides an informative perspective on recent developments in the theory of unitary group representations. An exciting feature of this book is the interesting interplay between measure theory, dynamical systems (ergodicity etc.) and von Neumann algebras, so that it can be recommended to anyone interested in these branches of mathematics and their connections with representation theory.

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