Preface

> "Il doit bien se présenter des problèmes de Physique mathématique pour lesquels les causes physiques de régularité ne suffisent pas à justifier les hypothèses de régularité faites lors de la mise en équation."[1]
>
> Jean Leray

This book is devoted to classical techniques in elliptic partial differential equations (PDEs), involving solutions that are not expected to be smooth. Some of the topics that are developed are: regularity theory, maximum principles, Perron–Remak method, sub- and supersolutions, and removable singularities. They rely on tools from measure theory, functional analysis, and Sobolev spaces, see e.g. [53], [125], [134], [160], and [345].

The goal is to investigate the *linear Dirichlet problem* involving the Laplacian:

$$\begin{cases} -\Delta u = \mu & \text{in } \Omega, \\ \quad\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{DP}$$

for an arbitrary *finite Borel measure* μ; we also consider the semilinear counterpart of problem (DP). The quantity $\mu(A)$ can be interpreted as the mass or total charge contained in a subset $A \subset \Omega$.

An example of solution is provided by the classical Green function with a Dirac mass $\mu = \delta_a$. For any smooth bounded open set Ω, this problem has a unique solution for any measure μ. By a solution, we mean a summable function that verifies the equation against smooth test functions vanishing on the boundary $\partial\Omega$, see [213]. This weak formulation implicitly encodes the zero boundary condition.

I have gathered several elegant proofs which are mostly available in the literature, but that are not necessarily widespread within the mathematical community. A surprising example is the simple argument leading to the fractional Sobolev embedding. I also explain the connection between trace inequalities and the strong approximation of diffuse measures.

The reader should feel free to choose a topic according to his/her own interests. The chapters have been conceived to be as independent as possible. In this respect, some redundancy is expected. I have also included review sections on measure theory

[1] "There must be problems in mathematical physics for which the physical causes of regularity do not suffice to justify the regularity assumptions made when the equation is formulated."

and Sobolev spaces. The exercises provide some complementary material, and are not intended to be solved in a first reading; their solutions can be found at the end of the book.

This project has originated from a set of lectures given at the Universidade Estadual de Campinas in 2005, and then from a full one-semester course in 2008. They were influenced by the enthusiasm and captivating style of H. Brezis, who had introduced me to these problems. In 2012, I deeply rewrote these notes, and the resulting monograph [288] won the Concours annuel in Mathematics of the Académie royale de Belgique. The text was then enlarged, including removable singularity principles and the Maz′ya–Adams trace inequalities. The notion of reduced measure, introduced with Brezis and Marcus [60] and pursued in [288], has been incorporated in the formalism of the nonlinear Perron–Remak method.

An updated list of corrections and misprints will be available in my personal website at `uclouvain.be`. The reader will find in the literature some recent advances on problems involving domains with little regularity [176], [236], [237], quasilinear operators in Euclidean spaces [165], [210], [218], or in metric spaces [29], Dirichlet problems involving trace measures on the boundary [226], connections to probability [120], [121], [200], [217], and trace inequalities [234], [239], that are not covered in this book.

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