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★**Bound states of the magnetic Schrödinger operator.**
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The mathematical study of the Schrödinger operator

$$-\Delta + V(x)$$

of quantum physics has become a vast undertaking that has helped spur the creation of modern mathematical analysis. The Schrödinger operator is the archetypical object whose discrete spectrum communicates a deep understanding into the underlying physics it represents. We refer to the spectra of the radiation absorbed or emitted by an atom that consists of discrete frequencies which suggests that electron shells (energy levels) only exist in a discrete set. Here, insight into the self-adjointness of operators allowed us to illuminate the photoelectric effect. Moreover, the spectral properties of the free Hamiltonian under certain perturbations may create discrete eigenvalues below the essential spectrum called bound states. Such bound states are important because they correspond to stationary solutions of the Schrödinger equation.

This book is aimed at an audience with some background in the spectral theory of Schrödinger operators [cf. M. C. Reed and B. Simon, *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*, Academic Press, New York, 1975; MR0493420]. The interested reader can also see [D. Krejčiřík, "Schrödinger operators and their spectra", course notes, BCAM Course on Applied and Computational Mathematics, Basque Cent. Appl. Math., 2010] for an introduction to bound states and waveguides.

The book under review opens with motivation for studying the magnetic Laplacian

$$(-ih\nabla + \mathbf{A})^2$$

which includes minimizers of the Ginzburg-Landau functional, and expressions for the n^{th} eigenvalue, $\lambda_n(h)$, by analogy to the electric Laplacian

$$-h^2\Delta + V(x),$$

where $h > 0$ is a (small) parameter related to Planck's constant. In the magnetic setting, the presence of the field $\mathbf{A}$ provides a force that is perpendicular to the velocity. Such an action tends to "bind" the particle, in addition, the magnetic field alone can give rise to bound states [see J. E. Avron, I. W. Herbst and B. Simon, Duke Math. J. **45** (1978), no. 4, 847–883; MR0518109]. Concerning the self-adjointness of the magnetic Laplacian and the associated magnetic Schrödinger operator, we refer to [T. Ikebe and T. Kato, Arch. Rational Mech. Anal. **9** (1962), 77–92; MR0142894]. Some of the earlier results concerning the magnetic Schrödinger operator in the semiclassical limit ($h \to 0$) appear in [J.-M. Combes, R. Schrader and R. Seiler, Ann Physics **111** (1978), no. 1, 1–18; MR0489509]. For a Lipschitz domain $\Omega \subset \mathbb{R}^d$ and $\mathbf{A} = (A_1, \ldots, A_d)$ a smooth vector (magnetic) potential on $\overline{\Omega}$, the *magnetic operator* is

$$\mathcal{L}_{h,\mathbf{A},\Omega} := \sum_{k=1}^{d} (-ih\partial_k + A_k)^2.$$

The operator obeys gauge invariance and is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$. The *magnetic Schrödinger operator* is

$$\mathcal{L}_{h,\mathbf{A},\Omega} + V(x)$$

for some suitable (electric) potential $V(x)$.

The focus of the book is on the discrete spectrum of the magnetic Schrödinger operator in the semiclassical limit. The development of this book moves away from the standard variational/min-max approach, whereby quasimodes are used as test functions for the quadratic form associated with the magnetic Schrödinger, but here a suitable microlocal representation of the operator is sought with a suitable application of the Spectral Theorem. For example, special attention is given to the estimation on the first eigenvalue $\lambda_1(h)$ by a magnetic well in the cases when the spatial dimension is two and three. Indeed, we quote from page 9, "the magnetic motion, in dimension three, can be decomposed into three elementary motions: the cyclotron motion, the oscillation along field lines, and the oscillation within the space of field lines. The concept of *magnetic harmonic approximation* developed in this book is an attempt to reveal, at the quantum level, these three motions in various geometric settings, without a deep understanding of the classical dynamics (one could call this a *semiquantum* approximation)." In addition, "we will provide the first examples of magnetic WKB constructions inspired by the recent work [V. Bonnaillie-Noël, F. Hérau and N. Raymond, Arch. Ration. Mech. Anal. **221** (2016), no. 2, 817–891; MR3488538]."

We remind the reader that in the standard analysis of the Schrödinger operator, the presence of a Hardy inequality is closely related to spectral threshold of the perturbed Hamiltonian. This fact serves as a sort of engine in the book. That is, there is a connection between the discrete spectrum and the presence of a magnetic field. These connections appear the proofs for the magnetic semiclassical asymptotic results and due to the recent literature [V. Bonnaillie-Noël and N. Raymond, Calc. Var. Partial Differential Equations **53** (2015), no. 1-2, 125–147; MR3336315; N. Dombrowski and N. Raymond, J. Spectr. Theory **3** (2013), no. 3, 423–464; MR3073418; N. Popoff and N. Raymond, SIAM J. Math. Anal. **45** (2013), no. 4, 2354–2395; MR3085117; N. Raymond, Comm. Partial Differential Equations **37** (2012), no. 9, 1528–1552; MR2969489; N. Raymond and S. Vũ Ngọc, Ann. Inst. Fourier (Grenoble) **65** (2015), no. 1, 137–169; MR3449150].

The book is divided into six parts. The first part, and most accessible part for an audience familiar with some functional analysis, consists of five chapters which are appropriately devoted to a review of spectral theory. These chapters also contain several important motivating examples. The discussion is oriented toward developing a more complete semiclassical analysis for the magnetic Schrödinger operator. In particular, Chapter 3 is dedicated to the electric Laplacian in dimension one. Part II presents the main theorems that are depended on later in the text. These include results on (i) spectral reductions, (ii) magnetic wells (dimension two), (iii) boundary magnetic wells (dimension three) and (iv) waveguides. Each of these four topics (i), (ii), (iii) and (iv) is then expounded in their own Parts III, IV, V and VI, respectively. This book represents the state of the art collection of recent results on the magnetic Schrödinger operator. *Joseph L. Shomberg*