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★**Metric measure geometry.**
Gromov's theory of convergence and concentration of metrics and measures.
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Among the many new and seminal ideas introduced in M. Gromov's by now classical *Metric structures for Riemannian and non-Riemannian spaces* [translated from the French, Progr. Math., 152, Birkhäuser Boston, Boston, MA, 1999; MR1699320], the concept of a metric geometry on the space of metric measure spaces proved to be especially fertile and inspiring during the decades following the book's publication. Unfortunately, as is often the case in Gromov's inspirational original texts, many of the details are omitted. Providing the detailed proofs represents the first goal of the book under review. Its second objective is to present a number of new, interesting developments, mainly concerning the connection, first developed in the author's joint work [K. Funano and T. Shioya, Geom. Funct. Anal. **23** (2013), no. 3, 888–936; MR3061776], between the fundamental ideas residing at the base of Gromov's work and a lower bound on (a fittingly generalized notion of) Ricci curvature.

Since Gromov's approach to the geometrization of metric measure spaces stemmed from the concentration of measure phenomena discovered by P. Levy and further developed by V. Milman, it is only natural that, after laying down some necessary background material in Chapter 1 in the book under review, the author focuses, in Chapter 2, on a concise yet clear presentation of the concentration phenomenon, which starts with an intuitive and essential characterization of the said phenomenon, namely the fact that any 1-Lipschitz continuous function is close to a constant on a domain of almost full measure. This is followed by the introduction and development of one of the central notions of Shioya's book, namely that of *observable distance* $d_{\mathrm{conc}}(X, Y)$ between two metric measure spaces X and Y, which can be roughly defined as being the difference between 1-Lipschitz functions on X and Y. (The full, technical definition of this notion is, however, deferred to Chapter 5 of the book.) The related notion of *separation distance*, as well as that of *Lipschitz order*, is also introduced. Beyond the basic results covered in this chapter, the relation between the kth eigenvalue of the Laplacian and the separation distance for a compact Riemannian manifold is also proved.

After an overview, in Chapter 3, of the basic notion of Hausdorff and Gromov-Hausdorff metrics, and the equivalence between the Gromov-Hausdorff and *distance matrices* of compact metric spaces is proven, the author concentrates in Chapter 4 on the so-called *box distance*. This proves to be a more elementary notion of distance between metric measure spaces than the concentration distance and, moreover, it is closely related to the established measured-Gromov-Hausdorff convergence of metric measure spaces. It is shown that concentration of metric measure spaces is equivalent to the convergence, in the box distance, of the associated pyramids, a *pyramid* being defined as a family of metric spaces that forms a directed set with respect to the Lipschitz order. The author proves that the Lipschitz order is stable under the box convergence and, furthermore, that any metric measure space can be approximated by a monotone nondecreasing sequence of finite-dimensional metric measure spaces. Moreover, the convergence of finite product spaces to infinite product spaces is also investigated (infinite products being a theme that is studied again, from a different perspective, in Chapter 7.2).

Chapter 5 introduces the notion of *measurement*, where the N-*measurement* of a metric measure space is defined as the set of push-forwards, under 1-Lipschitz maps, of the measure of a metric measure space to $\mathbb{R}^N$ endowed with the l_∞ norm. It turns out that measurements preserve the complete information of the given metric measure space but can more easily be studied than the original space itself. In particular, the concentration of metric measure spaces is equivalent to the convergence of the corresponding measurements, this being one of the main reasons for the study of pyramids' convergence.

Chapter 6 focuses, therefore, on pyramids: a metric on the space of pyramids is defined and its compactness is proven. (This development is done in parallel in the book under review and in the author's article ["Metric measure limits of spheres and complex projective spaces", preprint, arXiv:1402.0611].)

Chapter 7 contains the completion of the proof of one of the main results of the book, namely of the fact that the d_{conc}-completion of the space of metric measure spaces is embedded in the space of pyramids. Furthermore, the asymptotic concentration of finite product spaces is also studied, as is that of pyramids associated to the model spaces $S^n(\sqrt{n})$ and $\mathbb{R}^n(\gamma^n)$, where a sequence of metric measure spaces is said to *asymptotically concentrate* if it is a d_{conc}-Cauchy sequence. The connection between asymptotic and *spectral compactness*—see Definition 7.44 in the text—is also studied, and it is shown that any sequence of metric measure spaces that is both spectrally compact and asymptotic concentrates if the *observable diameter* is bounded from above.

Chapter 8 is dedicated to the notion of *dissipation*. A sequence of metric measure spaces is said to *δ-dissipate*, $\delta > 0$, iff any limit of associated pyramids contains all metric spaces with diameter $\leq \delta$. Furthermore, a sequence *infinitely dissipates* iff the associated pyramid converges to the space of metric measure spaces (a fact proven in Proposition 8.5). In contrast, the nondissipation theorem (namely Theorem 8.8) states that a sequence $\{F^n\}$ does not dissipate for any $\delta > 0$ if f is connected and locally connected. Its proof is based on an obstruction condition for dissipation. A number of interesting examples are also provided.

The concluding Chapter 9 is a presentation of the author's work in [K. Funano and T. Shioya, op. cit.] and it is dedicated to the connections between Gromov's ideas and the notion of generalized Ricci curvature bounded from below, as encapsulated by the *curvature-dimension* $CD(K, N)$, introduced by J. Lott and C. Villani [Ann. of Math. (2) **169** (2009), no. 3, 903–991; MR2480619] and K.-T. Sturm [Acta Math. **196** (2006), no. 1, 65–131; MR2237206; Acta Math. **196** (2006), no. 1, 133–177; MR2237207]. More precisely, the author proves the stability theorem of the curvature-dimension for concentration, and employs it to study the eigenvalues of the Laplacian on closed Riemannian manifolds. More specifically, under the condition of nonnegativity of Ricci curvature, the kth eigenvalue λ_k is dominated by a constant multiple C_k of the first eigenvalue λ_1, where C_k depends solely on the dimension of the manifold, which is independent of the dimension of the manifold. Furthermore, the stability of a lower bound for Alexandrov curvature under concentration is also proven.

In conclusion, as already stated in the opening paragraph of the present review, Shioya's book represents both a clarification of some of Gromov's original insights, as well as an exposition of novel and exciting results. As such, it represents an excellent introductory text for the interested researcher or graduate student. If anything, it is only a pity that for the sake of conciseness more geometric insights and illustrations that would have benefited especially the second type of relevant audience have been omitted. One can therefore hope that a textbook version, expanding the present book, will sometime follow.

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