

MR2768303 (2012d:30107) 30F10 01A55 14H55 30-01 30-03 30Fxx 32Q30  
de Saint-Gervais, Henri Paul

★Uniformisation des surfaces de Riemann. (French) [Uniformization of Riemann surfaces]

Retour sur un théorème centenaire. [A look back at a 100-year-old theorem]

The name of Henri Paul de Saint-Gervais covers a group composed of fifteen mathematicians : Aurélien Alvarez, Christophe Bavard, François Béguin, Nicolas Bergeron, Maxime Bourrigan, Bertrand Deroin, Sorin Dumitrescu, Charles Frances, Étienne Ghys, Antonin Guilloux, Frank Loray, Patrick Popescu-Pampu, Pierre Py, Bruno Sévenec, and Jean-Claude Sikorav.

*ENS Éditions, Lyon, 2010. 544 pp. €29.00. ISBN 978-2-84788-233-9*

The title of the book refers to the following theorem, called the *Uniformization theorem*: Any simply connected Riemann surface is biholomorphic to the Riemann sphere, or to the complex plane, or the unit disk in the complex plane.

Stated in this way, uniformization theory is a classification theory of Riemann surfaces. Originally, the term “uniformization” means the representation of a multi-valued complex function (defined as an algebraic curve, or as a solution of a differential equation, etc.) by a single-valued function of a complex variable (holomorphic or meromorphic). The above uniformization theorem, as stated, came as a solution to the uniformization problem of functions.

The uniformization theorem was proved in 1907, simultaneously by Köbe and by Poincaré. One would say, using a modern expression, that Poincaré and Köbe obtained their results “independently”, to stress the fact that it was not joint work and that they both should be given equal credit. But the term “independently” is in some sense not quite appropriate in this case, because the theorem has a very long history that includes results and attempts by several people, in which the major mathematicians of the nineteenth century were involved. These include, besides Köbe and Poincaré, Gauss, Abel, Jacobi, Riemann, Weierstrass, Clebsch, Fuchs, Schwarz, Klein, Fricke, Hilbert and Osgood, and there are certainly others. In this list, one should probably put Riemann’s name at the forefront.

The book under review is written by fifteen authors, who coined the name Henri Paul de Saint-Gervais to represent them: Henri Paul in honor of Henri Poincaré and Paul Köbe, and Saint-Gervais because the authors met for two weeks, one week in 2007 and one week in 2008, in Saint-Gervais-la-Forêt, a village in Sologne (France), for intensive work around this book. The book gives a comprehensive account of all the results that were obtained during the hundred years that preceded the proof of the uniformization theorem. It contains detailed proofs and an analysis of attempts of proofs, completing some of them and putting them in a modern perspective. Examples of special cases of the uniformization theorem include Riemann’s statement (known as the *Riemann mapping theorem*) that any simply connected open subset of the complex plane which is not equal to the whole plane is biholomorphic to the unit disc. The statement is contained in Riemann’s thesis (1851), and Riemann gave the major ideas for the proof. Riemann also made the assumption that the boundary of the set is piecewise smooth. Weierstrass discovered later on that Riemann’s proof, because of its dependence on the Dirichlet principle, was not complete. Among the major developments, one can mention the proof by Clebsch (1865) of the facts that every compact Riemann surface of genus zero is biholomorphic to the Riemann sphere and every compact Riemann surface of genus one is biholomorphic to the quotient of the plane by a lattice, and

Schwarz's explicit uniformization results of open sets whose boundaries are polygons (1896). Schwarz also discovered the so-called "Schwarzian derivative", which later on became one of the basic tools used by Poincaré in his approach to uniformization via differential equations. As another example of a major step in uniformization theory, one should mention Hilbert's formulation of this theory as one of his 23 problems (1900).

The uniformization theorem and its proof had a major impact on several fields of mathematics, including of course complex analysis, but also geometry, combinatorial group theory and topology. For instance the theory of covering surfaces, in particular the notion of universal cover, was developed as a tool for uniformization. Brouwer's theorem of invariance of domain was motivated by the so-called *continuity method*, which Klein, Poincaré and others used as a tool in the proof of various theorems around uniformization theory. The continuity method, on which the book under review expands, remained an important tool after the uniformization theorem was proved; it was used by Teichmüller, by Bers and by others in the moduli problem, which is a natural sequel to the uniformization problem. In group theory, one can mention the study of Poincaré polygons and their higher-dimensional analogues. There are also relations with physics, which were already highlighted by Riemann.

Uniformization theory, as we intend it today, is much broader than the theory that deals with the proof of the Köbe-Poincaré uniformization theorem. After this theorem was proved, uniformization theory went through several important developments. One can mention here Teichmüller's work on moduli and in particular his introduction of quasi-conformal mappings in that theory, the works of Ahlfors and Bers on the Beltrami equation, the so-called Bers "simultaneous uniformization theory", the uniformization theory of Kleinian groups, and the uniformization theory of higher-dimensional algebraic varieties, and there are many other developments. To show the importance of this living theory, let us quote L. Bers, from a survey he wrote on the uniformization problem [in *Mathematical developments arising from Hilbert problems (Proc. Sympos. Pure Math., Northern Illinois Univ., DeKalb, Ill., 1974)*, 559–609. Proc. Sympos. Pure Math., XXVIII, Amer. Math. Soc., Providence, RI, 1976; [MR0427623](#)]: "A significant mathematical problem, like the uniformization problem which appears as No. 22 on Hilbert's list, is never solved only once. Each generation of mathematicians, as if obeying Goethe's dictum *Was du ererbt von deinen Vätern hast, erwirb es, um es zu besitzen* ('That which you have inherited from your fathers, earn it in order to possess it'), rethinks and reworks solutions discovered by their predecessors, and fits these solutions into the current conceptual and notational framework. Because of this, proofs of important theorems become, as if by themselves, simpler and easier as time goes by—as Ahlfors observed in his 1938 lecture on uniformization. Also, and this is more important, one discovers that solved problems present further questions." Let us also quote W. Abikoff, from his 1981 survey [*Amer. Math. Monthly* **88** (1981), no. 8, 574–592; [MR0628026](#)]: "Research in uniformization theory has gone through several dormant periods since the concept of a (global) uniformization was introduced by Klein in 1882. Uniformization theorems of power unimaginable to the classical masters have been proved in the past twenty years and no end is in sight".

This is confirmed today, in 2011, where the subject is still under very active investigation.

The book under review is an excellent reference on this wonderful topic, and it should be a valuable reference for students and for researchers in Riemann surface theory, geometry and topology. (However, one should be aware of the fact that the authors are not historians and the book should not be taken as a strict historical reference.) One would hope for more books like this one.

*Athanase Papadopoulos*