Triebel, Hans * **1330.46003**
Hybrid function spaces, heat and Navier-Stokes equations.
EMS Tracts in Mathematics 24. Zürich: European Mathematical Society (EMS) (ISBN 978-3-03719-150-7/hbk). x, 185 p. EUR 48.00 (2015).

I refer to my review of the predecessor book [*H. Triebel*, Local function spaces, heat and Navier-Stokes equations. Zürich: European Mathematical Society (EMS) (2013; Zbl 1280.46-002)], where I mentioned that the main application of the local spaces introduced there was to local results for solutions of the heat and Navier-Stokes equations.

The Navier-Stokes equation, written as

$$\partial_t u + (u, \nabla u) - \Delta u + \nabla P = 0 \text{ in } \mathbb{R}^n \times (0, \infty),$$
$$\operatorname{div} u = \nabla \cdot u = 0 \text{ in } \mathbb{R}^n \times (0, \infty),$$
$$u(\cdot, 0) = u_0 \text{ in } \mathbb{R}^n,$$

with u an n-vector, is reduced to a nonlinear scalar heat equation by using the Leray projector, which is defined in terms of the Riesz transforms R_j,

$$\mathcal{P} f^k = f^k + R_k \sum_{j=1}^{n} R_j f^j,\ f = (f^1, \dots, f^n),$$

and the scalar equation is

$$\partial_t v - Dv^2(x,t) - \Delta v(x,t) = 0 \text{ in } \mathbb{R}^n \times (0, T),$$
$$v(\cdot, 0) = v_0 \text{ in } \mathbb{R}^n.$$

The author remarks that the theorems gave desired results in the global spaces $A^s_{p,q}(\mathbb{R}^n)$, but when applied to the local spaces $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$, he could not find boundedness theorems for the Leray projector and had to introduce the truncated Leray projector, enabling him to find solutions with the low frequencies cut off.

At the time, this appeared to be an artifact of the proof, but together with *M. Rosenthal* [Rev. Mat. Complut. 27, No. 1, 1–11 (2014; Zbl 1294.46038)], he has shown that it is impossible to extend the Riesz transform to the local Morrey space $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$, which means that the local truncation is indeed the best that one can do in solving the problems in local Morrey spaces.

In the present book, he introduces a class of hybrid spaces (hybrid because they are between the global spaces $A^s_{p,q}(\mathbb{R}^n)$ and the local spaces $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$). They are denoted $L^r A^s_{p,q}(\mathbb{R}^n)$ and have the property that the Riesz transforms map $L^r A^s_{p,q}(\mathbb{R}^n)$ into $L^r A^s_{p,q}(\mathbb{R}^n)$. The spaces are also multiplication algebras when $s + r > 0$ as were the local spaces.

The Morrey spaces $\mathcal{L}^r(\mathbb{R}^n)$ are defined by

$$\|f|\mathcal{L}^r(\mathbb{R}^n)\| = \sup_{J \in \mathbb{N}_0, M \in \mathbb{Z}^n} 2^{J(\frac{n}{p}+r)} \|f|L_p(Q_{J,M})\|,$$

where the $Q_{J,M}$ run over dyadic cubes in $\mathbb{R}^n$ on the integer lattice with lengths $2^{-J} \leq 1$ (hence local conditions). The global space $L^r(\mathbb{R}^n)$ is the same except that the sup is taken over dyadic cubes of all sizes ($J \in \mathbb{Z}$). In the final step with the original Morrey spaces, one further approximated f by polynomials, but already in $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$, the author originally used wavelet approximations as he does here,

$$\|f|L^r A^s_{p,q}(\mathbb{R}^n)\| = \sup_{jJ \in \mathbb{Z}, M \in \mathbb{Z}^n} 2^{J(\frac{n}{p}+r)} \inf_{g \in V^{\Psi}_{J,k+1}} \|f - g|L_p(Q_{J,M})\|,$$

where g is a suitable wavelet approximation of f.

The author uses these spaces in Chapter 4 to extend the results proved in his prior book for the nonlinear scalar heat equation above and the space $A^s_{p,q}(\mathbb{R}^n)$ to initial conditions in the spaces $L^r A^s_{p,q}(\mathbb{R}^n)$, $-\frac{n}{p} \leq r < \infty$. Since $A^s_{p,q}(\mathbb{R}^n) = L^{-n/p} A^s_{p,q}(\mathbb{R}^n)$, the generalization includes the prior result. The results are of a standard type. If the norm of the initial condition in $L^r A^s_{p,q}(\mathbb{R}^n)$ is small, the solution is bounded in $(0,1)$ with values in $L^r A^s_{p,q}(\mathbb{R}^n)$. If the initial condition has arbitrary size in $L^r A^s_{p,q}(\mathbb{R}^n)$, the solution is in a weighted $L^2(0,T)$ for some $T > 0$ with values in $L^r A^s_{p,q}(\mathbb{R}^n)$. The generalizations for the Navier-Stokes equation are given in Chapter 5. He concludes in Chapter 6 with a discussion of what can be said when the wavelet approximation is the Haar approximation. *Raymond Johnson (Columbia)*