

MR3308920 (Review) 46-02 35K05 35Q30 42B35 46E35 76D05
Triebel, Hans (D-FSU-MI)

★**Hybrid function spaces, heat and Navier-Stokes equations.**

EMS Tracts in Mathematics, 24.

European Mathematical Society (EMS), Zürich, 2014. x+185 pp.
ISBN 978-3-03719-150-7

This book is a culmination of a study of function spaces and the author applies his results to heat equations and Navier-Stokes equations. Before we discuss the contents of this book, let us recall some of the history of function spaces.

The classical L^p spaces, $1 \leq p \leq \infty$, are the most fundamental in harmonic analysis. An important advance in this direction was the introduction of Sobolev spaces in the late 1930's. In the 1950's, more and more attention was paid to the boundedness of singular integral operators on \mathbb{R}^n . Such singular integral operators fail to be L^∞ -bounded. As a substitute for the L^∞ -space, BMO spaces were invented.

In 1938, Morrey considered elliptic differential equations. In 1969, Peetre developed further Morrey's approach and introduced a normed space, which is now called the Morrey space and is a main concept in harmonic analysis.

Another important breakthrough in this direction was the introduction of Besov and Nikol'skiĭ spaces. Parallely, Triebel defined a class of spaces that have been called Triebel-Lizorkin spaces.

So far, we have mentioned four function spaces: BMO, Morrey spaces, Nikol'skiĭ-Besov spaces and Triebel-Lizorkin spaces. The aim of this book is to propose a framework including all of these function spaces and to apply it to the heat equation and the Navier-Stokes equation. The main focus is on the localized space $L^r A_{pq}^s(\mathbb{R}^n)$ described in (the short introductory) Chapter 1 of this book.

Let us describe the remaining parts of this book.

Chapter 2 deals with Morrey spaces. The local and global Morrey norms are defined as follows:

$$\|f\|_{\mathcal{L}_p^r(\mathbb{R}^n)} = \sup_{J \in \mathbb{N}_0, M \in \mathbb{Z}^n} 2^{J(\frac{n}{p}+r)} \|f\|_{L_p(Q_{J,M})},$$

and

$$\|f\|_{L_p^r(\mathbb{R}^n)} = \sup_{J \in \mathbb{Z}, M \in \mathbb{Z}^n} 2^{J(\frac{n}{p}+r)} \|f\|_{L_p(Q_{J,M})},$$

where $Q_{J,M} = 2^{-J}M + [0, 2^{-J})^n$. Here \mathcal{L} denotes an inhomogeneous space while L denotes a homogeneous one.

Morrey spaces are difficult to handle, as is described in Section 2.3.4. They are not separable nor do we have $D(\mathbb{R}^n)$ as a dense subspace of (local) Morrey spaces, as is written in Proposition 2.16. As we can see from Corollary 2.20, Morrey spaces are not reflexive. Keeping in mind the application to partial differential equations in the latter half of this book, the author considers the boundedness property of singular integral operators. He defines the predual spaces $\mathcal{H}^\rho L_p(\mathbb{R}^n)$ and $H^\rho L_p(\mathbb{R}^n)$. The definition originally follows from the paper by C. T. Zorko [Proc. Amer. Math. Soc. **98** (1986), no. 4, 586–592; [MR0861756](#)]. Sometimes Zorko spaces are referred to as block spaces. See Theorem 2.19 for the duality results. The reviewer would like to point out that the embedding results from Morrey spaces to power weighted Lebesgue spaces (Proposition 2.10 in the book under review) and from the Besov space $B_{1,1}^{\rho+n}(\mathbb{R}^n)$ to the Zorko block space $H^\rho L_p(\mathbb{R}^n)$ (Corollary 2.12 in the book under review) are quite useful.

In Chapter 6 the author presents an application of Morrey spaces and related function spaces to partial differential equations. Keeping this motivation in mind, he carefully defines singular integral operators in Section 2.5. In particular, the j -th Riesz transform is of interest. The Morrey space $L_p^r(\mathbb{R}^n)$ is strictly larger than the Lebesgue space $L^{-n/r}(\mathbb{R}^n)$ from Proposition 2.10. This part of the book closely follows the recent paper by M. Rosenthal and the author [Rev. Mat. t. **27** (2014), no. 1, 1–11; MR3149177].

In Section 2.6, the author considers the Haar wavelets and Morrey spaces. Theorem 2.31 gives a Haar wavelet characterization of Morrey spaces; it is in line with M. T. Lacey's 2007 work [Hokkaido Math. J. **36** (2007), no. 1, 175–191; MR2309828].

Chapter 3 is dedicated to the spatial local space $L^r A_{pq}^s$ and the special hybrid space $\mathcal{L}^r A_{pq}^s$. Let us recall the definitions. First, we let Ψ_F and Ψ_M be Daubechies wavelets. Note that Ψ_M satisfies the moment condition. The smoothness and the order of the moment are both assumed to be sufficiently high. By the routine procedure we can define the wavelet system

$$\{\Psi_m^{J^+} : m \in \mathbb{Z}^n\} \cup \{\Psi_{G,m}^j : j \geq J^+, G \in G^*, m \in \mathbb{Z}^n\}.$$

To define the hybrid space, we set

$$V_{J,u}^\Psi = \text{Span}(\Psi_m^J : m \in \mathbb{Z}^n), \quad J \geq 0, \quad V_{J,u}^\Psi = \{0\}.$$

Let $0 < p, q \leq \infty$, $s \in \mathbb{R}$ and $-n/p \leq r \leq \infty$. Let $u = k + 1$ where $k + 1$ satisfies

$$k + 1 > \max(s + \max(r, 0), n(1/p - 1) - s, -s)$$

in the case of $A = B$ and

$$k + 1 > \max(s + \max(r, 0), n(1/p - 1) - s, n(1/q - 1) - s, -s)$$

in the case of $A = F$. The space $L^r A_{p,q}^s(\mathbb{R}^n)_\Psi$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ for which

$$\|f|L^r A_{p,q}^s(\mathbb{R}^n)_\Psi\| = \sup_{(J,M) \in \mathbb{Z} \times \mathbb{Z}^n} 2^{J(\frac{n}{p} + r)} \inf_{g \in V_{J,k+1}^\Psi} \|f - g|A_{p,q}^s(\text{Int}(2Q_{J,M}))\|,$$

where $2Q_{J,M}$ is the double of $Q_{J,M}$, the cube concentric to $Q_{J,M}$ with volume 2^{n-J} . Likewise, the local space $\mathcal{L}^r A_{p,q}^s(\mathbb{R}^n)$ is defined by the norm

$$\|f|L^r A_{p,q}^s(\mathbb{R}^n)_\Psi\| = \|f|A_{p,q}^s(2Q_{0,M})\| + \sup_{(J,M) \in \mathbb{N} \times \mathbb{Z}^n} 2^{J(\frac{n}{p} + r)} \inf_{g \in V_{J,k+1}^\Psi} \|f - g|A_{p,q}^s(\text{Int}(2Q_{J,M}))\|.$$

The case when $p = \infty$ is excluded from the case $A = F$. We refer to [H. Triebel, *Theory of function spaces. II*, Monogr. Math., 84, Birkhäuser, Basel, 1992; MR1163193] for a definition of the function spaces on domains.

Let us describe some of the properties of the spaces just introduced.

1. $L^0 H_p^s(\mathbb{R}^n)$ is the same as $\text{bmo}^s(\mathbb{R}^n)$; see (3.193).
2. For the homogeneous counterpart $L^r \dot{A}_{p,q}^s(\mathbb{R}^n)$ one has

$$L^r A_{p,q}^s(\mathbb{R}^n) = L^r \dot{A}_{p,q}^s(\mathbb{R}^n) \cap L_p^r(\mathbb{R}^n),$$

as long as $r \in [-n/p, 0)$, as is proved in Theorem 3.50.

3. For some special parameters s , the space $L^r A_{p,q}^s(\mathbb{R}^n)$ admits a Morrey characterization.

Sequence spaces are introduced in Definition 3.24. Surprisingly enough, these notions have a lot to do with the spaces (see Definition 3.36) $A_{pq}^{s,\tau}$ defined by Yang and Yuan in [J. Funct. Anal. **255** (2008), no. 10, 2760–2809; MR2464191].

We refer to [W. Yuan, W. Sickel and D. C. Yang, *Morrey and Campanato meet Besov*,

Lizorkin and Triebel, Lecture Notes in Math., 2005, Springer, Berlin, 2010; [MR2683024](#) for more details on $A_{pq}^{s,\tau}$.

As a consequence of this fact, together with the results obtained in [Y. Sawano, D. C. Yang and W. Yuan, *J. Math. Anal. Appl.* **363** (2010), no. 1, 73–85; [MR2559042](#)], one concludes that the new space contains the Triebel-Lizorkin-Morrey spaces and, partially, Besov-Morrey spaces. Again, in the latter half of this chapter, the Haar wavelet characterization of these spaces is discussed. Most likely, a new trend in this theory will be to investigate function spaces having 0 smoothness. As an example of such an attempt, we refer to [O. V. Besov, *Mat. Sb.* **203** (2012), no. 8, 3–16; [MR3024810](#)].

Chapter 4 is short and deals with estimates for the heat semi-group and the non-linear heat equation

$$\partial_t u - \Delta u - Du^2 = 0$$

on the new function spaces described earlier.

Chapters 5 and 6 are devoted to the Navier-Stokes equation

$$\partial_t u - \Delta u + \mathbb{P}(u \otimes u) = 0 \quad \text{in } \mathbb{R}^n \times (0, T), \quad u(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^n.$$

When $T = \infty$, the solution is said to be time global. If T is finite, then the solution is said to be time local. After describing the notation briefly but neatly, the author transforms the equation into integral form. The aim is to consider solutions of the corresponding integral equation. When the initial data u_0 is small enough, then the equation is known to have a time global solution.

A key function space used in Chapters 5 and 6 is the Koch-Tataru space normed by

$$\|u\|_{E(\mathbb{R}^{n+1})} = \sup_{x \in \mathbb{R}^n} \sup_{t > 0} \sup_{j=1,2,\dots,n} \left(t |u^j(x, t)|^2 + t^{-n/2} \int_0^t \int_{|x-y| \leq \sqrt{t}} |u^j(y, t)|^2 dy d\tau \right)^{1/2},$$

where $u = (u^1, u^2, \dots, u^n)$ is a vector field of the function $x \in \mathbb{R}^n$ and $t > 0$. H. Koch and D. Tataru [*Adv. Math.* **157** (2001), no. 1, 22–35; [MR1808843](#)] showed that the solution is global and lies in E when the initial data has sufficiently small $\text{bmo}^{-1}(\mathbb{R}^n)_n$ -norm.

In Chapters 5 and 6 the author seeks a way to express the smallness of the data. One idea is to use the notion of infrared-damped data as defined in Definition 5.10. For example, u_0 is said to be infrared-damped if u_0 is orthogonal to Ψ_m , where Ψ_m is the wavelet system. Natural candidates here are the Haar wavelets, which are piecewise constant. The author considers the smallness of the data by using the Haar wavelet and he exhibits how to cope with the problem of having 0 smoothness. Another idea is to use Sobolev spaces with dominating mixed smoothness and to consider the space $S_p^1 W(\mathbb{R}^n, \alpha; \Gamma_{\alpha,p}^N)$.

From the viewpoint of function spaces, it may be worth noting that $\chi_{(0,1)^n} \in \dot{B}_{p,q}^s(\mathbb{R}^n)$ if q is finite and

$$n \left(\frac{1}{p} - 1 \right) < s < \frac{1}{p}$$

and that $\chi_{(0,1)^n} \in \dot{B}_{p,\infty}^s(\mathbb{R}^n)$ if

$$n \left(\frac{1}{p} - 1 \right) < s \leq \frac{1}{p}.$$

As an application of this fact, in Corollary 6.7 the author obtains an example of functions of BMO^{-1} .

Yoshihiro Sawano