Preface

This book is the continuation of [T13]. Our aim is twofold. First we develop the theory of *hybrid spaces* $L^r A^s_{p,q}(\mathbb{R}^n)$ which are between the nowadays well-known *global spaces* $A^s_{p,q}(\mathbb{R}^n)$ with $A \in \{B, F\}$ and their localization (or Morreyfication) $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$ as considered in detail in [T13]. Spaces $A^s_{p,q}(\mathbb{R}^n)$ cover (fractional) Sobolev spaces, (classical) Besov spaces and Hölder-Zygmund spaces, whereas local Morrey spaces $\mathcal{L}^r_p(\mathbb{R}^n)$ are special cases of the *local spaces* $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$. In [T13] we applied the theory of spaces $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$ to nonlinear heat equations and Navier-Stokes equations. But this caused some problems which will be discussed in the Introduction (Chapter 1) below. It came out quite recently that it is more natural in this context to switch from *local spaces* $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$ to *hybrid spaces* $L^r A^s_{p,q}(\mathbb{R}^n)$. This again will be illuminated in the Introduction below.

It is the second aim of this book to apply the theory of global spaces $A^s_{p,q}(\mathbb{R}^n)$ and hybrid spaces $L^r A^s_{p,q}(\mathbb{R}^n)$ to the Navier-Stokes equations

$$\partial_t u + (u, \nabla)u - \Delta u + \nabla P = 0 \qquad \text{in } \mathbb{R}^n \times (0, T), \tag{0.1}$$

$$\operatorname{div} u = 0 \qquad \text{in } \mathbb{R}^n \times (0, T), \tag{0.2}$$

$$u(\cdot, 0) = u_0 \qquad \text{in } \mathbb{R}^n, \tag{0.3}$$

in the version of

$$\partial_t u - \Delta u + \mathbb{P} \operatorname{div} (u \otimes u) = 0 \qquad \text{in } \mathbb{R}^n \times (0, T), \tag{0.4}$$

$$u(\cdot, 0) = u_0 \qquad \text{in } \mathbb{R}^n, \tag{0.5}$$

reduced to the scalar nonlinear heat equations

$$\partial_t v - D\, v^2 - \Delta v = 0 \qquad \text{in } \mathbb{R}^n \times (0, T), \tag{0.6}$$

$$v(\cdot, 0) = v_0 \qquad \text{in } \mathbb{R}^n, \tag{0.7}$$

where $0 < T \le \infty$. Here $u(x,t) = \big(u^1(x,t), \ldots, u^n(x,t)\big)$ in (0.1)–(0.5) is the unknown velocity and $P(x,t)$ the unknown (scalar) pressure, whereas $v(x,t)$ in (0.6), (0.7) is a scalar function, $2 \le n \in \mathbb{N}$. Recall $\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$ if $j = 1, \ldots, n$,

$$\big[(u, \nabla)u\big]^k = \sum_{j=1}^n u^j \partial_j u^k, \qquad k = 1, \ldots, n, \tag{0.8}$$

$$\operatorname{div} u = \sum_{j=1}^n \partial_j u^j, \qquad \nabla P = (\partial_1 P, \ldots, \partial_n P), \tag{0.9}$$

and by (0.2)

$$(u, \nabla)u = \operatorname{div} (u \otimes u), \qquad \operatorname{div} (u \otimes u)^k = \sum_{j=1}^n \partial_j (u^j u^k). \tag{0.10}$$

Furthermore, $\mathbb{P}$ is the Leray projector

$$(\mathbb{P}f)^k = f^k + R_k \sum_{j=1}^{n} R_j\, f^j, \qquad k = 1, \ldots, n, \tag{0.11}$$

based on the (scalar) Riesz transforms

$$R_k g(x) = i\Big(\frac{\xi_k}{|\xi|}\widehat{g}\Big)^{\vee}(x) = c_n \lim_{\varepsilon\downarrow 0} \int_{|y|\geq\varepsilon} \frac{y_k}{|y|^{n+1}}\, g(x-y)\,\mathrm{d}y, \quad x \in \mathbb{R}^n. \tag{0.12}$$

In (0.4), (0.5) there is no need to care about (0.2) any longer. But if, in addition, $\operatorname{div} u_0 = 0$ then $\operatorname{div} u = 0$ in our context (mild solutions based on fixed point assertions). In the scalar equation (0.6) we used the abbreviation

$$D f = \sum_{j=1}^{n} \partial_j f. \tag{0.13}$$

As mentioned above we dealt in [T13] with the above equations in the context of the global spaces $A^s_{p,q}(\mathbb{R}^n)$. Rigorous reduction of (0.1)–(0.3) to (0.4), (0.5) and finally to (0.6), (0.7) requires a detailed study of the nonlinearity $u \mapsto u^2$ and of boundedness of Riesz transforms in the underlying spaces. In [T13] we tried to extend this theory to some local spaces $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$. But one needs some modifications, especially a replacement of the Riesz transforms by some truncated Riesz transforms. In the Introduction below we repeat the above considerations in greater details and discuss in particular this somewhat disturbing (but unavoidable) point. The hybrid spaces $L^r A^s_{p,q}(\mathbb{R}^n)$ preserve many desirable properties of the local spaces $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$ but avoid the above-indicated shortcomings. They are between global spaces and local spaces, which may justify calling them hybrid spaces. They coincide with the well-studied spaces $A^{s,\tau}_{p,q}(\mathbb{R}^n)$, $\tau = \frac{1}{p} + \frac{r}{n}$, including the global spaces $A^{s,0}_{p,q}(\mathbb{R}^n) = A^s_{p,q}(\mathbb{R}^n)$ as special cases.

Chapter 1 is the announced Introduction where we return to the above description in greater details and with some references. Chapter 2 deals with local and global Morrey spaces $\mathcal{L}^r_p(\mathbb{R}^n)$, $L^r_p(\mathbb{R}^n)$, their duals and preduals and, in particular, with the question whether the Riesz transforms R_k in (0.12) are bounded maps in these spaces and what they look like. This chapter is self-contained and we hope that it is of interest for researchers in this field. In Chapter 3 we develop the theory of the hybrid spaces $L^r A^s_{p,q}(\mathbb{R}^n)$ as needed for our above-outlined purposes. It comes out that many basic properties for the local spaces $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$ can be transferred easily from [T13] to the hybrid spaces $L^r A^s_{p,q}(\mathbb{R}^n)$. We concentrate on some new aspects which will be crucial in the context described above. Similarly we carry over and complement in Chapter 4 the theory of heat equations in the global spaces $A^s_{p,q}(\mathbb{R}^n)$ as developed in [T13] to the hybrid spaces $L^r A^s_{p,q}(\mathbb{R}^n)$. Chapter 5 deals with Navier-Stokes equations especially in the version (0.4), (0.5) in hybrid spaces. Then one is in a rather comfortable position, clipping together related assertions of the two preceding chapters. But again we add a few new aspects. In particular, if the admitted initial data are infrared-damped then the related local solutions of the Navier-Stokes equations can be extended globally in time. These considerations will be continued in Chapter 6 now specified to the global spaces $A^s_{p,q}(\mathbb{R}^n)$ and extended to the spaces $S^r_{p,q}A(\mathbb{R}^n)$ with dominating mixed smoothness. We discuss conditions for the initial data in terms of Haar wavelets, Faber bases, and sampling in connection with the hyperbolic cross, ensuring solutions of the Navier-Stokes equations which are global in time. Furthermore we add some comments about the influence of large Reynolds numbers. This chapter is largely independent of the preceding considerations.

We assume that the reader has a working knowledge about basic assertions for the spaces $A^s_{p,q}(\mathbb{R}^n)$. But to make this book independently readable we provide related notation, facts, and detailed references. Formulae are numbered within chapters. Furthermore in each chapter all definitions, theorems, propositions, corollaries and remarks are jointly and consecutively numbered. References are ordered by names, not by labels, which roughly coincide, but may occasionally cause minor deviations. The bracketed numbers following the items in the Bibliography mark the page(s) where the corresponding entry is quoted. All unimportant positive constants will be denoted by c (with additional marks if there are several c's in the same formula). To avoid any misunderstanding we fix our use of $\sim$ (equivalence) as follows. Let I be an arbitrary index set. Then

$$a_i \sim b_i \quad \text{for} \quad i \in I \qquad \text{(equivalence)} \tag{0.14}$$

for two sets of positive numbers $\{a_i : i \in I\}$ and $\{b_i : i \in I\}$ means that there are two positive numbers c_1 and c_2 such that

$$c_1\, a_i \le b_i \le c_2\, a_i \qquad \text{for all } i \in I.$$