

MR3236321 (Review) 51-01 14N05 51A05 51Nxx
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★**Analytic projective geometry.**

EMS Textbooks in Mathematics.

European Mathematical Society (EMS), Zürich, 2014. xvi+620 pp.
ISBN 978-3-03719-138-5

The book under review is devoted to giving an analytic presentation of projective geometry over the real and complex fields, together with their affine and metric specializations. This presentation is strongly based on linear algebra. In accordance with usual conventions, but for a few exceptions (such as rational normal curves) the study is limited to considering linear and quadratic geometric objects, since higher degree ones belong rather to algebraic geometry. The core of the projective part is the study of linear varieties, cross-ratio, projective transformations and quadric hypersurfaces, including the projective classification of the latter. In real projective geometry, imaginary points need to be considered as soon as non-linear equations appear; they are introduced by a formal construction of the complex extension of a real projective space. The basic objects allowing the application of projective geometry to the affine and metric geometries—projective closure of an affine space, improper hyperplane, absolute quadric—are introduced and used to reformulate the basic elements of the affine and metric geometries in projective terms. In particular, affine and metric classifications of quadric hypersurfaces are presented as successive specializations of the projective classification.

The book comprises 11 chapters and 2 appendices. A more detailed description follows.

Chapter 1 introduces the main objects of study in this book, namely the (finite-dimensional) projective spaces over a field, which is almost always either the complex numbers or the real numbers. After introducing and defining the projective space, the first properties and objects are given: linear varieties, projectivities, the classical theorems of Pappus and Desargues, projections, sections and perspectivities.

Chapter 2 introduces the projective (or homogeneous) coordinates: this is done starting—as usual—by projective references, then the definition of projective coordinates is given, followed by their first properties and related definitions, such as the change of coordinates. These concepts are applied to the definitions given in Chapter 1, e.g. implicit and parametric equations of linear varieties, incidence relations with coordinates, matrices of a projectivity, etc. The chapter ends with the study of the cross-ratio of four points of \mathbb{P}^1 and harmonic sets of four points of \mathbb{P}^1 .

Chapter 3 deals with affine geometry: after a quick introduction of affine spaces and their basic properties, the author moves on to show how projective spaces arise naturally as extensions of affine spaces via their projective closure, adjoining the points at infinity. Then, it is shown how the basic affine notions can be interpreted in a projective frame. The chapter ends with a presentation of F. Klein's Erlangen Program.

Chapter 4 goes back to projective geometry and gives an introduction to projective duality. The core of the chapter is clearly the principle of duality, which is presented after the basic definitions and results about dual projective spaces. Projective coordinates, dual of a projectivity and biduality are presented as well.

Chapter 5 is about projective transformations. First of all, it is explained how the complex projective space is a natural extension of the real projective space. Then, projective transformations of lines are analysed in great detail; in the general case,

involutions, collineations, correlations, perspectivities, and (singular) projectivities are studied.

In Chapter 6 the author starts the study of quadrics; after giving the notion of projective quadric and proving the first elementary statements, he recasts some results about projective transformations of lines in terms of quadrics in \mathbb{P}^1 . Then, he proceeds to give the usual projective notions associated to quadrics, such as conjugation, polarity, degenerate quadrics and cones, quadric envelopes, etc.

Chapter 7 deals with the classification of quadrics, both projective and affine, in the real and the complex case. Further properties connected with the classification are presented as well.

Chapter 8 contains the most important results concerning the projective generation and internal structure of projective conics and quadric surfaces. These have a large number of consequences that include many affine and metric specializations. In more detail, the chapter starts with Steiner's theorem together with its consequences; then, from the fact that a smooth conic is isomorphic to \mathbb{P}^1 , many consequences are deduced, such as Pascal's and Brianchon's theorems. Then, quadrics in \mathbb{P}^3 are studied from the point of view of the lines contained in them; in the same spirit, the Klein quadric and its geometry are also considered.

Chapter 9, on projective spaces of quadrics, moves closer to (classical) algebraic geometry: after defining the notions of rational curve and effective divisor on \mathbb{P}^1 , it proceeds with linear systems of quadrics. As is to be expected, the case of pencils is considered first, with special emphasis on the pencils of conics; Desargues' theorem on pencils of quadrics is also proven. Then, to relate quadrics and quadric envelopes, the notion of apolarity is introduced. Finally, there is an introduction to rational normal curves, and in particular to the twisted cubic.

Chapter 10 presents the metric properties and metric classification of quadrics. After studying circles and spheres, metric classification and properties of the smooth conics are given, including a treatment of their foci. Then, the author moves on to quadric surfaces, which are studied in detail. Finally, the general theory is presented, first by determining the metric reduced equations of the quadrics, then by finding their metric invariants, culminating in the metric classification of general quadrics.

The more technical projective classifications of collineations, pencils of quadrics and correlations, together with the algebraic background they require, are the contents of the last chapter, i.e. Chapter 11.

Two less common applications are presented in two appendices: one goes back to the origins of projective geometry by showing the projective foundations of the practical rules of perspective; the other explains a parametric form of Klein's model of Euclidean and non-Euclidean plane geometries.

A number of exercises are placed at the end of each chapter; some of them are nice classical results, not central enough to have a place in the text. Results proved in the exercises are used in other exercises, but not in the text.

The material in this book is suitable for courses in projective geometry for undergraduate students with a working knowledge of a standard first course in linear algebra. The text is a valuable guide to graduate students and researchers working in areas using or related to projective geometry, such as algebraic geometry and computer vision, and to anyone wishing to gain an advanced view on geometry as a whole. *Pietro De Poi*