Introduction

Two of the most influential advances in the history of geometry occurred in France in the short period 1636–1639. One was the introduction and systematic use of coordinates conceived, independently, by R. Descartes (*La Géometrie*, 1637) and P. Fermat (*Ad locos planos and solidos isagoge*, around 1636). Coordinates allowed the use of resources from algebra and analysis in the foundations and development of geometry, which is usually called *analytic geometry*. Analytic geometry is thus not a part of geometry, but rather a method that applies to all parts of geometry; probably calling it *analytic method in geometry* would have been better. The other was the first study of the properties of figures invariant by perspectivities, due to G. Desargues (*Brouillon project d'une atteinte aux évenements des rencontres d'un cone avec un plan*, 1639). A perspectivity between two different planes of ordinary three-space consists in fixing a point O outside the planes and taking two points, one on each plane, as correspondent if and only if they are collinear with O. Perspectivities were already used in the studies of perspective originated by Renaissance painting and in the construction of sundials (Desargues himself wrote on both subjects). Also known to Desargues was the work of Apollonius of Perga (262–190 BC), who implicitly used perspectivities to relate arbitrary sections of a circular cone to one of its circular sections. The properties invariant by perspectivities are called *projective*; they are satisfied by large classes of figures. The study of projective properties, and of the classes of figures satisfying them, is the subject of *projective geometry*, which thus leaves aside notions such as parallelism and distance, which are not invariant by perspectivities.

Analytic and projective geometry provided two quite different ways to overcome the lack of generality inherent in the methods of the ancient Greek geometers. They both boosted geometry to a long period of continued progress in which the whole field of geometry was widely enlarged and far better understood. Analytic geometry had an immediate success: since the publication of *La Géometrie* till the end of the eighteenth century, there was a constant flourishment almost entirely based in the use of coordinates. Meanwhile, shadowed by analytic geometry, projective geometry made little progress. It was not until the beginning of the nineteenth century that interest in properties invariant by perspectivities was renewed: then the pioneering work of Desargues and his few followers was largely extended to subjects such as duality, cross-ratio and polarity. Cartesian coordinates being not well suited to it, in a first stage projective geometry was developed without using coordinates, which was called *pure* or *synthetic* (method in) geometry. After a while, the introduction of homogeneous coordinates allowed the fruitful application of the analytic method to projective geometry. All together, projective geometry emerged in its own right as an important – and very nice – part of geometry. However, the most important fact was that while projective geometry was growing, it was gradually realized that each of its notions and theorems had as specializations a number of already known non-projective notions or theorems; for instance, both the ratio of distances between three aligned points (*affine ratio*) and the angle between two lines could be expressed in terms of a more general projective invariant called *cross ratio*. Eventually it became clear that this situation was quite general: all the geometry known at the time could be seen as a specialization of projective geometry. This allowed a new presentation in which projective geometry was taken as the fundamentals of geometry, and provided a better and deeper understanding of the entire geometrical field, by unveiling the projective common roots of apparently unrelated parts of geometry.

During the second half of the nineteenth century, projective geometry gradually merged with the theory of algebraic functions giving rise to a specialized branch of geometry called *algebraic geometry*. An important part of it, named *projective algebraic geometry*, is devoted to the study of *projective algebraic varieties*, which are figures of projective spaces defined by polynomial equations. Very active at the research level today, projective algebraic geometry may be understood as the natural continuation of projective geometry.

The development of computer vision in recent years has brought a renewed interest in projective geometry and, especially, in its metric applications. Computer vision starts from the same objects that were at the basis of Desargues's work: the perspectivities, renamed *pinhole cameras*. Its basic goal is the analysis and reconstruction of three-dimensional scenes from perspective images of them. So it is not surprising to find many notions and results of classical projective geometry playing today a central role in computer vision.

This book is devoted to giving an analytic presentation, strongly based on linear algebra, of what may be considered to be the whole of n-dimensional projective geometry over the real and complex fields, together with their affine and metric specializations. When relevant, the specifics of the low-dimensional cases $n = 1, 2, 3$ are also dealt with in detail. According to the usual conventions, but for a few exceptions, we will limit ourselves to considering linear and quadratic geometric objects, the study of the higher degree ones belonging rather to algebraic geometry. The core of the projective part is the study of linear varieties, cross ratio, projective transformations and quadric hypersurfaces, including the projective classification of the latter. Special attention is paid to the projective structures on sets of certain geometric objects, such as hyperplanes or quadrics: considering these structures multiplies the applications of the abstract projective theorems and makes one of the main differences between old and modern geometry. In real projective geometry, imaginary points need to be considered as soon as non-linear equations do appear; they are introduced by a formal construction of the complex extension of a real projective space. The basic objects allowing the application of projective geometry to the affine and metric geometries – projective closure of an affine space, improper hyperplane, absolute quadric – are introduced and used to reformulate the basic elements of the affine and metric geometries in projective terms. In particular the affine and metric classifications of quadric hypersurfaces are presented as successive specializations of their projective classification. Since, besides their intrinsic interest, the affine and metric applications are very good illustrations of the projective results they come from, they are presented as soon as there is enough projective material to support them. The more technical projective classifications of collineations, pencils of quadrics and correlations, together with the algebraic background they require, are the contents of the last chapter. Two less usual applications are presented in two appendices: one goes back to the origins of projective geometry by showing the projective foundations of the practical rules of perspective; the other explains a parametric form of Klein's model of Euclidean and non-Euclidean plane geometries. A number of exercises are proposed at the end of each chapter; some of them are nice classical results, not central enough to have a place in the text. Results proved in the exercises are used in other exercises, but not in the text.

As already said, both the foundations and the general development of this book are analytic. When suitable, short synthetic arguments have been used at some points. The more elementary parts of projective geometry may be given purely synthetic presentations of which there are nice examples in the literature. These presentations proceed by direct and very clever proofs, and are the best for discussing foundations, as they start from a short number of axioms. However, it is quite unrealistic – and certainly non-practical – to try to cover the whole contents of this book by the exclusive use of the synthetic method: the more advanced parts of projective geometry cannot be developed without algebraic support and many applications need to use coordinates; this is notably the case for all the applications to computer vision, which run by numerical computation. Further, and more decisive, it makes no sense to hide the powerful underlying algebraic structures that are the basis of the analytical treatment, as they are so important as points and lines. The fecund relationship between algebraic and geometric structures alone boosted the continued progress which during the last one-hundred and fifty years and going far beyond the scope of this book, caused algebra, arithmetic and algebraic geometry to appear today as simply different views of a unique, rich and deep, field of knowledge.

Hopefully this book will be useful to undergraduate students taking a course on projective geometry, to graduate students and researchers working in fields that make use of projective geometry – such as algebraic geometry or computer vision – and also to anyone wishing to gain an advanced view on the whole of the geometrical field. All of them will find here a fairly complete account of what has been classically considered to be projective geometry and applications, written in a modern language and accordingly to today's standards of rigour. There are included a number of topics for which there is a lack of suitable references. Many of them, such as singular projectivities, correlations, Plücker coordinates, line-complexes and twisted cubics, are of use in computer vision.

Obviously, the contents of this book largely exceeds what is reasonable to teach in a single course, and many different courses may be given using parts of it. I have given introductory courses based on Chapters 1 and 2, selected parts of Chapters 3 to 6 and the projective and affine classifications of quadrics from Chapter 7. I have also given more advanced courses including most of Chapter 8, part of Chapter 9, the first four sections of Chapter 10 and one or both of the Appendices A, B.

Requirements for reading this book are the contents of a standard course on linear algebra, including vector spaces, linear maps, matrices, diagonalization, linear forms, dual space and bilinear forms, as well as some knowledge of the usual vectorial presentations of the linear affine and metric geometries, their basics being, however, quickly recalled in the text. Although not strictly needed, some background on the classic Euclidean presentation of the elementary geometry of lines, polygons, circles, etc., is advisable for a better understanding of the projective presentations of some of these subjects shown here.

I took my first – and very inspiring – course on projective geometry from Professors J. B. Sancho Guimerá and J. M. Ortega Aramburu, back in 1967: most of the spirit of that course is still present in this book. Other important sources have been classical Italian books, especially E. Bertini's *Introduzione alla Geometria Proiettiva degli Iperspazi* ([2]) and G. Castellnuovo's *Lezioni di Geometria Analitica* ([4]), the excellent *Algebraic Projective Geometry* by J. P. Semple and G. T. Kneebone ([25]) and, finally, O. Schreier and E. Sperner's *Projective Geometry of n dimensions* ([24]), which systematically deals with the n-dimensional case and makes intensive use of linear algebra.

After teaching courses on projective geometry for about thirty-five years, I am very grateful to all colleagues – too many to be named – who either taught parallel courses or collaborated in my own courses: I learned from all of them and hence they all have positively contributed to this book.

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