The defocusing NLS equation and its normal form.

The book contains a comprehensive study of the defocusing nonlinear Schrödinger equation (dNLS) on a circle

\[ i\partial_t u = -\partial_x^2 u + 2|u|^2 u, \quad u(x + 1, t) = u(x, t) \quad \text{for all } x \in \mathbb{R} \text{ and } t \in \mathbb{R}, \]

as a Hamiltonian system. The defocusing case is specific by the fact that the time-evolution operator in the Lax pair is skew-adjoint; dNLS defines an isospectral deformation in the space of all potentials \( \varphi = (\varphi_1, \varphi_2) \) such that \( \varphi_1 = u, \varphi_2 = \pi \) (real type potentials) of the Zakharov-Shabat (ZS) linear problem, and thus a decomposition of the space into isospectral sets; the spectra are real as the ZS operator is self-adjoint.

The authors' goal is to provide a complete and self-contained description of the canonical structure of dNLS. They construct a global coordinate system in which the equation is a classical integrable Hamiltonian system with infinitely many degrees of freedom.

The idea is to show a path to the solution of the dNLS by construction of the fundamental solution. For this purpose the authors build a Poisson structure and a canonical transformation to the action-angle variables \( I_n, \theta_n, n \in \mathbb{Z} \). Finally, they express the system in terms of the Birkhoff coordinates \( x_n = \sqrt{2} I_n \cos \theta_n, \quad y_n = \sqrt{2} I_n \sin \theta_n \), that allows them to obtain global analyticity results. The book contains results of the authors since Grébert’s thèse doctorat of 1990 [Problèmes spectraux inversés pour les systèmes akns sur la droite réelle, Univ. Paris-Nord]. A contribution of J. Pöschel is mentioned in the preface.

The main result is the existence of a bi-analytic diffeomorphism to a unique global real-analytic system of coordinates (the Birkhoff coordinates), such that the dNLS Hamiltonian is a function of the actions alone (contrary to the focusing NLS which does not admit global Birkhoff coordinates). It is shown that KAM theory holds for the system.

The introductory part of the book consists of a preface, table of contents, list of figures and a nice overview of the ideas and results. Of the four chapters (20 sections altogether), the first two have preparatory character, while the main results are in chapters 3 and 4. The core is followed by six appendices. Of the nine figures, one illustrates topology of isospectral sets; the remaining ones visualize properties of the ZS spectrum on the complex plane. The book has 48 references up to 2014. The last two pages contain a list of almost all non-standard symbols used in the book (an exception: a blackboard bold bracket is defined in the text only on page 53).

Chapter 1 contains a proof of the existence and uniqueness of a matrix-valued solution to the initial value problem and an outline of the basic facts about spectra of the ZS operators. The latter include estimations of the basic components used later in the final construction, in particular the asymptotics as the spectral parameter tends to infinity.

In Chapter 2 more properties of the spectra of the ZS operators are proven. Characteristic functions are found whose zeros are the Dirichlet, Neumann and periodic spectra (the distinction is based on various forms of the periodic boundary conditions imposed on the components of the two-component potential \( \varphi \)). A proof is given that
the spectrum of the ZS operator is a pure point and consists of an unbounded bi-infinite sequence of periodic eigenvalues; the spectra are analytic functions of the potential values (locally). Explicit formulae for their gradients are given. Furthermore, it is proven that for the potentials of real type (the defocusing case) the eigenvalues come in pairs and each of them is a compact function in the $L^2_r$ space of the two-component real-type potentials. The intervals between the eigenvalues of a pair (‘spectral gaps’) are open. Isospectral sets are proven to be compact NLS-invariant tori, which may be parameterized by the positions of the Dirichlet eigenvalues and the sign of the anti-discriminant of the fundamental solution. Finally, the chapter contains a definition of Poisson’s brackets in terms of the Wronskian and a construction of real canonical conjugate variables to the Dirichlet eigenvalues with a discussion of their analytic and asymptotic properties.

The last two chapters contain the main construction of the action-angle variables (Liouville coordinates) and the Birkhoff coordinates. In Chapter 3, construction of the action-angle variables by means of Arnold’s formula is performed as in the classical finite-dimensional case. It yields explicit formulae for the action variables in terms of the discriminant and its derivative with respect to the eigenvalue, as well as an explicit formula for the angles as the respective action-derivatives of the standard symplectic form. The method requires extension of the $L^2_r$ space of the real-type potentials to its small complex neighborhood in the corresponding complex space $L^2_c$. The actions are real analytic on $L^2_r$, while the angles are defined on a dense open domain in $L^2_r$ and become real analytic when considered mod $\pi$.

In Chapter 4 the main result is a theorem stating that the map of $L^2_r$ to the Birkhoff coordinates is real analytic and extends to an analytic map in $L^2_c$. The mapping is shown to preserve the canonical structure. Finally, the map is shown to be a global real analytic diffeomorphism, which maps each isospectral set one-to-one onto the torus in the space of two-component real sequences. The Hamiltonian is a real analytic function of the actions alone in those coordinates.

In addition to its particular goal—description of the dNLS as a Hamiltonian system—the book is an excellent example of the construction of the action-angle and Birkhoff coordinates for a highly nontrivial infinite-dimensional integrable system. The book is very clearly written. The outline of ideas presented at the beginning of chapters 1, 3, and 4 makes the fairly complex topics easily understandable, while the main thread is woven with full precision and exactness. The six appendices: on properties of analytic maps between complex Banach spaces (A), on the Hamiltonian formalism and Poisson structure (B), on infinite products (C), on estimations of the Fourier coefficients (D), on multiplicities of eigenvalues (E) and another one (F) containing several lemmas specific for the needs of the book, make the book available for graduate students and non-specialists.

Piotr P. Goldstein

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