

# Contents

Preface	vii
<b>21 Examples of Multivariate Approximation</b>	<b>1</b>
21.1 Multivariate Approximation	6
21.2 Infinitely Differentiable Functions	10
21.3 Gaussian Kernel Functions	16
21.4 Monotone Functions	24
21.5 Convex Functions	30
21.6 Notes and Remarks	35
<b>22 Randomized Setting: Multivariate Approximation</b>	<b>37</b>
22.1 Multivariate Approximation for the Class $\Lambda^{\text{all}}$	41
22.2 Multivariate Approximation for the Class $\Lambda^{\text{std}}$	45
22.3 Relations between Minimal Errors, Part I	47
22.4 Weak Tractability	52
22.4.1 Example: Unweighted Tensor Products	55
22.5 Relations between Minimal Errors, Part II	57
22.6 Constructive and Non-Constructive Bounds	67
22.7 Exponential Convergence	72
22.8 Polynomial Tractability	76
22.8.1 Example: Unweighted Tensor Products (Continued)	80
22.9 Quasi-Polynomial Tractability	81
22.9.1 Example: Unweighted Tensor Products (Continued)	87
22.10 Approximation for Weighted Korobov Spaces	87
22.10.1 Results for the Normalized Error Criterion	92
22.10.2 Results for the Absolute Error Criterion	94
22.10.3 Implementation Cost of the Algorithm $A_{n,k}$	95
22.11 Notes and Remarks	95
<b>23 Randomized Setting: Linear Problems</b>	<b>99</b>
23.1 Linear Multivariate Problems	104
23.1.1 Quasi-Polynomial Tractability for $\Lambda^{\text{all}}$	106
23.1.2 Lower Bounds	112
23.2 Linear Multivariate Problems over $L_{2,Q_d}$	114
23.2.1 Lower Bounds for the Class $\Lambda^{\text{std}}$	115
23.2.2 Upper Bounds for the Class $\Lambda^{\text{std}}$	118
23.2.3 Finite Dimensional Case	119
23.2.4 Infinite Dimensional Case	123
23.3 Linear Multivariate Problems for More General $F_d$	132
23.3.1 Examples	134

23.3.2	Relations between Minimal Errors . . . . .	139
23.3.3	Polynomial Order of Convergence . . . . .	142
23.3.4	Examples (Continued) . . . . .	145
23.3.5	Tractability of $S = \{S_d\}$ . . . . .	147
23.4	Multivariate Integration . . . . .	155
23.5	Tensor Product Linear Functionals . . . . .	162
23.5.1	Decomposable Kernels . . . . .	163
23.5.2	Example: Integration of Smooth Functions . . . . .	167
23.5.3	Example: Centered Discrepancy . . . . .	171
23.5.4	Non-Decomposable Kernels . . . . .	172
23.6	Notes and Remarks . . . . .	180
<b>24</b>	<b>Average Case Setting: Multivariate Approximation</b>	<b>185</b>
24.1	Linear Multivariate Problems for $\Lambda^{\text{all}}$ . . . . .	187
24.1.1	Quasi-Polynomial Tractability for $\Lambda^{\text{all}}$ . . . . .	191
24.2	Multivariate Approximation for $\Lambda^{\text{std}}$ . . . . .	197
24.2.1	Example: Wiener Measure . . . . .	197
24.3	Relations between Minimal Errors . . . . .	201
24.3.1	Exponential Convergence . . . . .	205
24.4	Weak Tractability . . . . .	207
24.4.1	Example: Unweighted Tensor Products . . . . .	209
24.5	Polynomial Tractability . . . . .	210
24.5.1	Example: Unweighted Tensor Products (Continued) . . . . .	213
24.6	Quasi-Polynomial Tractability . . . . .	214
24.6.1	Example: Unweighted Tensor Products (Continued) . . . . .	216
24.7	Approximation for Weighted Korobov Spaces . . . . .	217
24.8	Euler and Wiener Integrated Processes . . . . .	221
24.8.1	Euler Integrated Process . . . . .	222
24.8.2	Wiener Integrated Process . . . . .	226
24.8.3	Increased Smoothness . . . . .	228
24.9	Notes and Remarks . . . . .	230
<b>25</b>	<b>Average Case Setting: Linear Problems</b>	<b>232</b>
25.1	Linear Multivariate Problems . . . . .	234
25.1.1	Examples . . . . .	236
25.2	Relations between Minimal Errors . . . . .	238
25.2.1	Examples (Continued) . . . . .	241
25.2.2	Examples (Continued) . . . . .	247
25.3	Relations to Worst Case . . . . .	250
25.3.1	Curse of Dimensionality for $S_d$ . . . . .	255
25.4	Finite-Order Weights . . . . .	258
25.4.1	Problem Formulation . . . . .	259
25.4.2	Algorithms for $\Lambda^{\text{std}}$ . . . . .	264
25.4.3	Final Remarks . . . . .	277

25.5	Notes and Remarks . . . . .	280
<b>26</b>	<b>Worst Case Setting: Multivariate Approximation</b>	<b>282</b>
26.1	Multivariate Approximation for the Class $\Lambda^{\text{all}}$ . . . . .	286
26.1.1	Example: Arbitrary Sequence $e^{\text{wor}}(n, d; \Lambda^{\text{all}})$ . . . . .	287
26.2	Multivariate Approximation for the Class $\Lambda^{\text{std}}$ . . . . .	289
26.3	Relations between Minimal Errors: Infinite Trace . . . . .	292
26.3.1	Proof of Theorem 26.1 . . . . .	294
26.4	Relations between Minimal Errors: Finite Trace, Part I . . . . .	304
26.4.1	Tractability . . . . .	309
26.5	Relations between Minimal Errors: Finite Trace, Part II . . . . .	318
26.6	Relations between Minimal Errors: Finite Trace, Part III . . . . .	325
26.6.1	Speed of Convergence and Tractability . . . . .	329
26.7	Exponential Convergence . . . . .	336
26.8	Notes and Remarks . . . . .	337
<b>27</b>	<b>Worst Case Setting: Linear Problems</b>	<b>340</b>
27.1	Linear Multivariate Problems . . . . .	343
27.2	Relations to Multivariate Approximation, Part I . . . . .	344
27.3	Relations to Multivariate Approximation, Part II . . . . .	353
27.3.1	Polynomial Order of Convergence . . . . .	356
27.3.2	Tractability of $S = \{S_d\}$ . . . . .	358
27.4	Finite-Order Weights . . . . .	362
27.4.1	Basic Properties . . . . .	364
27.4.2	Auxiliary Results . . . . .	367
27.4.3	Upper Bounds on $n^{\text{wor}}(\varepsilon, S_d; \Lambda)$ . . . . .	370
27.4.4	Lower Bounds on $n^{\text{wor}}(\varepsilon, S_d, \Lambda)$ . . . . .	377
27.4.5	Multivariate Integration . . . . .	379
27.5	Finite-Order Weights: Algorithms for $\Lambda^{\text{std}}$ . . . . .	381
27.5.1	Algorithms for $\Lambda^{\text{std}}$ . . . . .	383
27.5.2	Extensions . . . . .	394
27.5.3	Applications . . . . .	399
27.6	WTP Algorithms for Product Weights . . . . .	406
27.6.1	Formulation of the Problem . . . . .	406
27.6.2	Weighted Tensor Product Algorithms . . . . .	413
27.6.3	The Class $\Lambda^{\text{all}}$ . . . . .	417
27.6.4	The Class $\Lambda^{\text{std}}$ . . . . .	425
27.6.5	Applications . . . . .	436
27.7	Notes and Remarks . . . . .	441
<b>28</b>	<b>Nonlinear Problems</b>	<b>443</b>
28.1	Quasilinear Problems: General Results . . . . .	444
28.1.1	Problem Formulation . . . . .	446
28.1.2	Quasilinear Problems . . . . .	449

28.1.3	Tensor Products and General Weights . . . . .	452
28.1.4	Some Results for Multivariate Approximation . . . . .	454
28.1.5	Results for Quasilinear Problems . . . . .	455
28.2	The Poisson Equation . . . . .	463
28.2.1	The Dirichlet Problem . . . . .	470
28.2.2	The Neumann Problem . . . . .	482
28.3	Fredholm Equations of the Second Kind . . . . .	494
28.3.1	Basic Concepts . . . . .	497
28.3.2	Fredholm and Multivariate Approximation . . . . .	500
28.3.3	Some Examples . . . . .	510
28.3.4	Weighted Tensor Product Spaces . . . . .	518
28.3.5	Interpolatory Algorithms for Tensor Product Spaces . . . . .	521
28.4	Further Examples . . . . .	528
28.4.1	The Heat Equation . . . . .	528
28.4.2	The Helmholtz Equation . . . . .	529
28.4.3	Multivariate Approximation and Nonlinear Problems . . . . .	529
28.4.4	Approximation as a Nonlinear Problem: Non-Convex Classes . . . . .	530
28.5	Notes and Remarks . . . . .	532
<b>29</b>	<b>Power of Function Values for Multivariate Approximation</b>	<b>534</b>
29.1	Worst Case Setting . . . . .	537
29.1.1	Double Hilbert Case . . . . .	539
29.1.2	Single Hilbert Case . . . . .	544
29.1.3	Banach Case . . . . .	547
29.2	Randomized Setting . . . . .	552
29.2.1	Double Hilbert Case . . . . .	554
29.2.2	Other Cases . . . . .	555
29.3	Average Case Setting with Gaussian Measures . . . . .	556
29.4	Notes and Remarks . . . . .	558
<b>F</b>	<b>List of Open Problems</b>	<b>559</b>
<b>G</b>	<b>Errata for Volumes I and II</b>	<b>569</b>
	Bibliography	573
	Index	583