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★**Frobenius algebras. I.**

Basic representation theory.

EMS Textbooks in Mathematics.

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The book by Skowroński and Yamagata is the first of two volumes devoted to the representation theory of Frobenius algebras. The aim of the first volume is to provide a comprehensive introduction to the representation theory of finite-dimensional associative algebras with a view towards Frobenius algebras. The second volume will be more advanced in using covering techniques to reduce the representation theory of Frobenius algebras to the representation theory of algebras of finite global dimension and then being able to apply tilting theory. The first volume, however, is primarily addressed to graduate students as well as to non-specialists. This volume is self-contained and the only prerequisite is a basic knowledge of linear algebra. The authors include detailed proofs for most of the results. Furthermore, many examples interspersed throughout the book as well as about 50 exercises at the end of each of the first four chapters on the general theory and about 30 exercises at the end of each of the last two chapters on examples of Frobenius algebras will make it easier to understand the theory.

The notion of a Frobenius algebra has its origin in two papers, “Theorie der hyperkomplexen Grössen I, II”, written by Ferdinand Georg Frobenius at the very beginning of the twentieth century [Berl. Ber. **1903**, 504–537, 634–645; JFM 34.0238.02]. More than thirty years later, at the end of the 1930’s, Brauer, Nesbitt, and Nakayama realized the significance of the class of algebras studied by Frobenius for understanding non-semisimple algebras and their modules, and started to investigate Frobenius algebras intensively. There are several reasons for which Frobenius algebras are important in the representation theory of finite-dimensional associative algebras. First of all, many interesting classes of finite-dimensional associative algebras are Frobenius algebras, namely, group algebras of finite groups; Hecke algebras of finite reflection groups, Brauer tree algebras, and other algebras defined combinatorially; certain quotients of universal enveloping algebras of finite-dimensional Lie algebras in prime characteristic governing the representations of the latter; small half-quantum and quantum groups; Nichols algebras; and any finite-dimensional Hopf algebra. As modules over Frobenius algebras are either projective (= injective) or have infinite projective dimensions, Frobenius algebras have features quite different from those of other classes of algebras appearing in representation theory. For example, hereditary algebras, which occur naturally as path algebras of finite acyclic quivers, have global dimension at most one and so the properties of their modules are controlled by Ext^1 . Contrary to this, higher extensions play an important role for Frobenius algebras. Very often some total cohomology of a Frobenius algebra is finitely generated and then many properties of its modules can be studied by applying tools from algebraic geometry to certain homogeneous affine varieties associated to each module. Finally, several prominent conjectures in the representation theory of finite-dimensional associative algebras can be reduced to self-injective algebras, and the module category of every finite-dimensional self-injective algebra is equivalent to the module category of a Frobenius algebra.

The book can be divided into three parts, which we are going to describe now in some detail. The first part consists of the first three chapters and makes up about half of the

book. It deals with concepts and techniques that are important in the representation theory of arbitrary finite-dimensional associative algebras. In Chapter I basic notions and results relevant to representation theory are collected. Finite-dimensional hereditary algebras are introduced and several properties of their modules are proved. In particular, finite-dimensional hereditary algebras are characterized as path algebras of finite acyclic quivers. Chapter II is dedicated to a proof of the Morita equivalence and Morita-Azumaya duality theorems, and the authors explain the necessary concepts in sufficient detail. In Chapter III the reader is introduced to Auslander-Reiten theory of irreducible morphisms and almost split sequences, which is fundamental to modern representation theory, providing useful combinatorial and homological invariants of module categories.

The second part is Chapter IV (the longest chapter in the book, making up for about 160 pages), which is an introduction to finite-dimensional self-injective algebras and the properties of their module categories. The chapter begins with the classical result of Frobenius from 1903 and goes on to the early results due to Brauer, Nesbitt, and Nakayama. Symmetric algebras, weakly symmetric algebras, and Frobenius algebras are introduced, and it is shown that in this order each type of algebra is an example of the next, and that Frobenius algebras are self-injective. As already mentioned above, every finite-dimensional self-injective algebra is Morita equivalent to a Frobenius algebra. The Nakayama automorphism of a Frobenius algebra is defined and the relationship between the Nakayama functor and the Nakayama automorphism is explained. Then the authors define the syzygy functor for any finite-dimensional associative algebra and prove that the Auslander-Reiten translation of a finite-dimensional self-injective algebra is the composition of the Nakayama functor and the square of the syzygy functor (in any order). Moreover, periodic modules and periodic algebras are introduced, and it is shown that every non-projective module over a finite-dimensional self-injective algebra of finite representation type is periodic. It is mentioned without proof that over algebraically closed ground fields one has the stronger result that finite-dimensional self-injective algebras of finite representation type are periodic algebras. The authors also define higher extension spaces of modules and prove that the extension algebra of a finite-dimensional periodic indecomposable module over a finite-dimensional self-injective algebra modulo nilpotent homogeneous elements is a polynomial algebra in one variable, and the degree of the variable equals the period of the module. Many examples of Frobenius algebras are introduced along the way, namely, semisimple algebras, group algebras of finite groups, trivial extension algebras of arbitrary finite-dimensional associative algebras, certain quadratic quantum algebras, Brauer tree algebras, and canonical mesh algebras of Dynkin type. Chapter IV concludes with a proof of the Riedtmann-Todorov theorem on the shape of the stable Auslander-Reiten quiver of a finite-dimensional self-injective algebra of finite representation type.

The third part of the book consists of two chapters and is devoted to two important classes of Frobenius algebras. Chapter V is the shortest chapter, at about 50 pages, and in it the authors classify the finite reflection groups and then associate to any such group and a nonzero element q of a field a finite-dimensional algebra over this field, the Hecke algebra of the reflection group. It is shown that such a Hecke algebra is always a symmetric algebra which specializes for $q = 1$ to the group algebra of the reflection group. In the last chapter, Chapter VI, the authors develop the basic theory of finite-dimensional Hopf algebras from scratch and prove that every finite-dimensional Hopf algebra is a Frobenius algebra with a Nakayama automorphism of finite order. The first part follows from the Larson-Sweedler structure theorem for Hopf modules and the second part is a consequence of the Fischman-Montgomery-Schneider formula for the Nakayama automorphism of a finite-dimensional Hopf algebra in conjunction with Radford's theorem on the finite order of the antipode of a finite-dimensional Hopf

algebra. Also in this chapter several examples of Hopf algebras are considered, namely, certain truncated polynomial algebras, restricted universal enveloping algebras, and Taft algebras.

The book by Skowroński and Yamagata is very well written and will be especially useful for graduate students and non-experts because of the detailed exposition and the wealth of examples and exercises, as well as the inclusion of more advanced results in order to bridge the gap to the research literature on the subject and to put the discussed material into perspective. My only quibble about this book is that an index of symbols is missing and would have been very helpful for the prospective reader. *Jörg Feldvoss*

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