

Introduction

The major concern of this book is the representation theory of finite dimensional associative algebras with an identity over a field. In simplest terms, it is an approach to the problem of describing how a finite number of linear transformations can act simultaneously on a finite dimensional vector space over a field. The representation theory of finite dimensional algebras traces its origin to the middle part of the nineteenth century with Hamilton's discovery of the quaternions, the first noncommutative field, and investigations of finite groups via their representations in matrix algebras over the field of complex numbers. The main achievements of the representation theory of algebras of the latter part of the nineteenth and the beginning of the twentieth century concerned the structure of semisimple finite dimensional algebras over fields and their representations. A new and fundamental view on the representation theory of finite dimensional algebras over fields came in the 1930s from the papers by Noether who gave the theory its modern setting by interpreting representations as modules. The module theoretical approach allowed one to apply in the representation theory of finite dimensional algebras the language and techniques of category theory and homological algebra. The modern representation theory of finite dimensional algebras over fields can be regarded as the study of the categories of their finite dimensional modules and the associated combinatorial and homological invariants.

A prominent role in the representation theory of finite dimensional algebras over fields is played by the Frobenius algebras. This is a wide class of algebras containing the semisimple algebras, blocks of group algebras of finite groups, the Hecke algebras of finite Coxeter groups, the finite dimensional Hopf algebras, and the orbit algebras of the repetitive categories of algebras. The Frobenius algebras have their origin in the 1903 papers by Frobenius who discovered that the left and right regular representations of a finite dimensional algebra over a field, defined in terms of structure constants with respect to a fixed linear basis, are equivalent if and only if there is a very special invertible matrix intertwining both representations. Later Brauer, Nesbitt and Nakayama realized that the study of finite dimensional algebras with the property that the left and right regular representations are equivalent is crucial for a better understanding of the structure of nonsemisimple algebras and their modules, and called them Frobenius algebras. In a series of papers from 1937–1941, Brauer, Nesbitt and Nakayama established characterizations of Frobenius algebras which were independent of the choice of a linear basis of the algebra. In particular, we may say that a finite dimensional algebra A over a field K is a Frobenius algebra if there exists a nondegenerate K -bilinear form $(-, -): A \times A \rightarrow K$ which is associative, in the sense that $(ab, c) = (a, bc)$ for all elements a, b, c of A . Moreover, if such a nondegenerate, associative, K -bilinear form is symmetric, A is called a symmetric algebra. We also mention that the Frobenius algebras are selfinjective

algebras (projective and injective modules coincide), and the module category of every finite dimensional selfinjective algebra over a field is equivalent to the module category of a Frobenius algebra.

The main aim of the book is to provide a comprehensive introduction to the representation theory of finite dimensional algebras over fields, via the representation theory of Frobenius algebras. The book is primarily addressed to graduate students starting research in the representation theory of algebras as well as mathematicians working in other fields. It is hoped that the book will provide a friendly access to the representation theory of finite dimensional algebras as the only prerequisite is a basic knowledge of linear algebra. We present complete proofs of all results exhibited in the book. Moreover, a rich supply of examples and exercises will help the reader to understand the theory presented in the book.

We divide the book into two volumes.

The aim of the first volume of the book is two fold. Firstly, it serves as a general introduction to basic results and techniques of the modern representation theory of finite dimensional algebras over fields, with special attention to the representation theory of Frobenius algebras. The second aim is to exhibit prominent classes of Frobenius algebras, or more generally selfinjective algebras.

The first volume of the book is divided into six chapters, each of which is subdivided into sections. We start with Chapter I presenting background on the finite dimensional algebras over a field and their finite dimensional modules. In Chapter II we present the Morita equivalences and the Morita–Azumaya dualities for the module categories of finite dimensional algebras over fields. Chapter III is devoted to presenting background on the Auslander–Reiten theory of irreducible homomorphisms and almost split sequences, and the associated combinatorial and homological invariants. Chapter IV forms the heart of the first volume of the book and contains fundamental classical and recent results concerning the selfinjective algebras and their module categories. In Chapter V we present the classification of finite reflection groups of real Euclidean spaces via the associated Coxeter graphs and show that they provide a wide class of symmetric algebras over an arbitrary field, called the Hecke algebras. In the final Chapter VI we describe the basic theory of finite dimensional Hopf algebras over fields and show that they form a distinguished class of Frobenius algebras for which the Nakayama automorphisms are of finite order.

The main aim of the second volume of the book, “Frobenius Algebras II. Orbit Algebras” is to study the Frobenius algebras as the orbit algebras of repetitive categories of finite dimensional algebras with respect to actions of admissible automorphism groups. In particular, we will introduce covering techniques which frequently allow us to reduce the representation theory of Frobenius algebras to the representation theory of algebras of small homological dimension. A prominent role in these investigations will be played by tilting theory and the authors theory of selfinjective algebras with deforming ideals.

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