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★**Nonlinear potential theory on metric spaces.**

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European Mathematical Society (EMS), Zürich, 2011. *xii*+403 *pp.*
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The book under review is a nice introduction to the theory of upper gradient–based Sobolev-type function spaces on metric measure spaces. It is written by two experts in the field of potential theory in the metric setting.

Given a function u on a metric space X, a non-negative Borel measurable function g on X is said to be an upper gradient of u if, whenever γ is a compact curve of finite length in X, the inequality

$$|u(y) - u(x)| \le \int_\gamma g\, ds$$

holds; here, x and y denote the end points of γ. The notion of upper gradients was first proposed by J. Heinonen and P. Koskela [Acta Math. **181** (1998), no. 1, 1–61; MR1654771], who used the term "generalized gradients" rather than "upper gradients". The notion of upper gradients was used by the reviewer in [Rev. Mat. Iberoamericana **16** (2000), no. 2, 243–279; MR1809341] to propose an analog of a Sobolev-type function class in the setting of metric measure spaces; such an analog is referred to as Newtonian spaces, in recognition of the fact that the upper gradient property is an extension of the classical Fundamental Theorem of Calculus. In this book, the authors give an exposition of the theory of upper gradients, and use this theory to study the potential theory of p-energy minimizers on doubling metric measure spaces that support a Poincaré-type inequality with respect to the upper gradients. A measure μ on a metric space X is said to be doubling if μ is a Borel measure and there is a constant $C \ge 1$ such that, whenever $x \in X$ and $r > 0$, we have $\mu(B(x, 2r)) \le C\,\mu(B(x, r))$; and a metric measure space is said to support a p-Poincaré inequality if there are constants $C > 0$ and $\lambda \ge 1$ such that, whenever $B(x, r)$ is a ball in X and u is a function on X with upper gradient g, we have

$$\inf_{c\in\mathbb{R}} \int_{B(x,r)} |u - c|\, d\mu \le C\, r\, \frac{\mu(B(x, r))}{\mu(B(x, \lambda r))^{1/p}} \left(\int_{B(x,\lambda r)} g^p\, d\mu \right)^{1/p}.$$

The book is conceived as a graduate textbook on nonlinear potential theory related to the notion of upper gradients, is written in a style accessible to graduate students familiar with measure theory and topology, and includes a lot of results developed by the authors and their collaborators. In particular, the Perron solutions to the p-energy minimizing problem for domains in such a metric setting are studied in depth. More specifically, the discussions on the regular vs irregular boundary and the Wiener criterion are quite extensive.

Chapter 1 of the book gives an introduction to upper gradients and the Newtonian spaces, and in Chapter 2 the properties of upper gradients and weak upper gradients (a slightly relaxed version of the upper gradients) are studied. The reader who wishes to get an introduction to the subject of Newtonian spaces and upper gradients will find the discussion in Chapter 2 extremely useful, especially the part pertaining to the gluing lemma (Section 2.4). Chapter 2 also provides a discussion on the existence of minimal weak upper gradients of functions that have an upper gradient in the class $L^p(X)$.

Chapter 4 is devoted to the study of spaces supporting a p-Poincaré inequality, although the discussion of Keith's characterizations of spaces supporting a p-Poincaré inequality is not extensive [however, see S. Keith and X. Zhong, Ann. of Math. (2) **167** (2008), no. 2, 575–599; MR2415381; S. Keith and K. Rajala, Math. Scand. **95** (2004), no. 2, 299–304; MR2098359; S. Keith, Math. Z. **245** (2003), no. 2, 255–292; MR2013501; and references therein]. Furthermore, the geometric characterizations of certain Poincaré inequalities, due to Heinonen and Koskela in [New Zealand J. Math. **28** (1999), no. 1, 37–42; MR1691958; op. cit.; MR1654771], are not treated extensively in the book, since the principal focus of the book is to develop the tools essential for the study of potential theory in the metric setting. On the other hand, a nice discussion on the relationship between the enlargement factor λ in the Poincaré inequality and the quasiconvexity constant of the metric space can be found in Section 4.8 of the book; the reviewer is not aware of anywhere else in the literature that this relationship is studied. Chapter 4 also gives a proof of the fact that a complete metric space, equipped with a doubling measure supporting a p-Poincaré inequality for some $1 \leq p < \infty$, must be quasiconvex (that is, each pair of points in the space can be connected by a curve of length at most a constant multiple of the distance between the two points).

Chapter 5 studies fine properties of Newtonian spaces of functions on metric measure spaces where the measure is doubling and supports a Poincaré inequality, and Chapter 6 gives a good exposition on capacities related to such Newtonian spaces.

The study of nonlinear potential theory begins in Chapter 7 of the book. In Chapter 7 the authors introduce p-superminimizers and the related obstacle problem, and study the (interior) regularity of p-harmonic functions. In Chapter 8 they show that given a domain in a metric space equipped with a doubling measure supporting a p-Poincaré inequality, a function that is a p-energy minimizer can be modified on a set of capacity zero to obtain a Hölder continuous version. A Harnack inequality is also discussed in this chapter. Chapter 10 is devoted to the study of the Dirichlet problem for p-harmonic functions, and a discussion of regular boundary points is also contained in this chapter. The Perron method of constructing solutions to the Dirichlet problem is discussed, and issues of resolutivity of boundary data are extensively described. The discussion of regular boundary points is continued in Chapter 11, where barrier characterizations and the Wiener criterion are described. The reviewer notes that a discussion on the boundary Harnack principle and a discussion on the propagation of the modulus of continuity of boundary data are both lacking in this book [however, see J. L. Lewis and K. Nyström, Ann. Sci. École Norm. Sup. (4) **40** (2007), no. 5, 765–813; MR2382861; J. L. Lewis, N. L. P. Lundström and K. Nyström, in *Perspectives in partial differential equations, harmonic analysis and applications*, 229–266, Proc. Sympos. Pure Math., 79, Amer. Math. Soc., Providence, RI, 2008; MR2500495; J. L. Lewis and K. Nyström, Ann. of Math. (2) **172** (2010), no. 3, 1907–1948; MR2726103; H. Aikawa and N. Shanmugalingam, Michigan Math. J. **53** (2005), no. 1, 165–188; MR2125540; J. Differential Equations **220** (2006), no. 1, 18–45; MR2182078; H. Aikawa et al., Potential Anal. **26** (2007), no. 3, 281–301; MR2286038; I. Holopainen, N. Shanmugalingam and J. T. Tyson, in *Papers on analysis*, 147–168, Rep. Univ. Jyväskylä Dep. Math. Stat., 83, Univ. Jyväskylä, Jyväskylä, 2001; MR1886620].

The discussion of nonlinear potential theory is continued in Chapters 12 and 13, where removable singularities and classification (principle of trichotomy) of irregular boundary points are discussed.

The book, while not exhaustive in its coverage of nonlinear potential theory in the metric setting, does contain the essential tools necessary for one who wishes to venture into this area of research. It is eminently readable, and for the most part complete details are provided. The reviewer finds the discussion on regular and irregular boundary points especially appealing. Although the discussion in the book does not extend to the study of p-quasiminimizers, as in [J. Kinnunen and N. Shanmugalingam, Manuscripta Math. **105** (2001), no. 3, 401–423; MR1856619], the book does provide extensive notes and references that fill in the gap. *Nageswari Shanmugalingam*

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