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★**Efficient numerical methods for non-local operators.**
$\mathcal{H}^2$-matrix compression, algorithms and analysis.
EMS Tracts in Mathematics, 14.
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The monograph under review is devoted to presenting an overview of theoretical results and practical algorithms for working with $\mathcal{H}^2$-matrices, as announced in its subtitle: "$\mathcal{H}^2$-matrix compression, algorithms and analysis". Besides the Lecture Notes on Hierarchical Matrices from the Max-Planck-Institute from Leipzig and the book of M. Bebendorf [*Hierarchical matrices*, Lect. Notes Comput. Sci. Eng., 63, Springer, Berlin, 2008; MR2451321], as far as I know, there has not previously been a monograph devoted to this topic. While the book of Bebendorf focuses on $\mathcal{H}$-matrices, the present monograph is about $\mathcal{H}^2$-matrices. For this reason alone, the author, coming from the powerful school on hierarchical matrices of W. Hackbusch, should be thanked and congratulated.

Hierarchical matrices ($\mathcal{H}$-matrices) are a class of usual matrices which are data-sparse, but they allow one to work with them in an efficient way. The efficiency refers to the possibility of representing large-scale fully populated matrices with almost linear (linear or logarithmic-linear) complexity, for arbitrary precision. $\mathcal{H}$-matrices not only approximate matrices arising in many important scientific computing problems, they also offer a set of matrix arithmetic operations like evaluation, multiplication, factorization and inversion. This fact will be exploited in order to set up approximate preconditioners in a convenient way. $\mathcal{H}^2$-matrices introduce an additional hierarchical structure to reduce the storage requirements and computational complexity of $\mathcal{H}$-matrices.

In the Foreword section, the author offers suggestions to different types of audiences as to what chapters they should read (e.g., if they are students of numerical mathematics, or if interested in using $\mathcal{H}$-matrices to treat integral equations, or if they want to apply them to elliptic partial equations). Also, if one is interested in this topic, there is an HLib software package, which is free of charge for research purposes.

This monograph is structured in ten chapters.

Chapter 1 is devoted to presenting the origin of $\mathcal{H}^2$-matrices, describing which kind of matrices are prone to be compressed, which kind of operations are able to be treated efficiently, which problems can be solved efficiently, and finally giving us the organization of the book.

Chapter 2 shows the basic concepts of hierarchical matrices and $\mathcal{H}^2$-matrices, using as a model problem the one-dimensional integral operator. In the end, the author presents tables for the approximation errors for the model problem and the storage requirements per degree of freedom for $\mathcal{H}^2$-matrix approximations and standard array representation for the model problem, respectively.

Chapter 3 is concerned with the generalizations of the definition of $\mathcal{H}^2$-matrices to the multi-dimensional setting. For this, Section 3.1 introduces the general definition of a cluster tree and proves a number of its basic properties; the next subsection contains the general definition of a block cluster tree; Section 3.3 describes a simple geometrical construction for cluster and block cluster trees; Section 3.4 gives the general definition of hierarchical matrices; the next subsection introduces general cluster bases and the basic concepts for estimating the complexity of algorithms for $\mathcal{H}^2$-matrices; Section 3.6

is devoted to the general definition of $\mathcal{H}^2$-matrices and $\mathcal{H}^2$-matrix spaces and to proving bounds for the storage complexity. Section 3.7 presents the most important algorithm in the context of $\mathcal{H}^2$-matrices: the evaluation of the product of an $\mathcal{H}^2$-matrix and an arbitrary vector; Section 3.8 contains a number of definitions that allow one to express complexity estimates in terms of matrix dimension instead of number of clusters.

Chapter 4 is devoted to an application of the theory presented in Chapter 3, to a class of problems in which densely populated matrices occur naturally, namely the numerical treatment of integral operators. For doing this, the author finds a separable approximation of the kernel function underlying the integral operator, and this approximation gives rise to an $\mathcal{H}^2$-matrix approximation of the corresponding stiffness matrix. A separable approximation can be obtained via Taylor expansion (Section 4.3), or via interpolation by a polynomial (Section 4.4). Error analysis in both one- and multi-dimensional cases is also provided.

Chapter 5 presents results related to orthogonal cluster bases and matrix projections. Section 5.1 introduces the concept of orthogonal cluster bases and provides us with simple algorithms for checking whether a given cluster basis is orthogonal. Section 5.2 contains algorithms for converting dense $\mathcal{H}$-matrices into $\mathcal{H}^2$-matrices using orthogonal cluster bases; the next subsection defines cluster operators, and Section 5.4 is devoted to an algorithm that turns an arbitrary nested cluster basis into an orthogonal nested cluster basis in optimal complexity. A variant of the former algorithm is depicted in Section 5.5. Section 5.6 contains a collection of methods for computing the error introduced by a projection into an $\mathcal{H}^2$-matrix space. The chapter is concluded with numerical experiments that prove that the bounds for the complexity of the new introduced algorithms are optimal and that the adaptive error control works as predicted by theory.

In Chapter 6, the author addresses the following questions: what are the basic structural properties of $\mathcal{H}^2$-matrices, which types of matrices can be approximated by $\mathcal{H}^2$-matrices, and if a matrix is approximated, how can one find that approximation efficiently?

Chapter 7 is about a priori matrix arithmetic. Once a matrix has been approximated by an $\mathcal{H}^2$-matrix, the question of solving corresponding systems of linear equations has to be answered. The author is presenting here methods for solving this system that can be performed using matrix-matrix products. In the last three subsections, by using the orthogonal projections introduced in Chapter 5, it is proved how the best approximation of a matrix-matrix product in a given $\mathcal{H}^2$-matrix space can be computed very efficiently.

Chapter 8 is about a posteriori matrix arithmetic. The algorithms introduced in the previous chapter compute the best approximation of the matrix-matrix product in a given space. But if this space is not chosen correctly, the resulting error can be quite large. Here is presented an alternative algorithm that constructs an $\mathcal{H}^2$-matrix approximation of the matrix-matrix product and chooses the cluster bases in such a way that a given precision can be guaranteed.

Chapter 9 presents an application to an elliptic partial differential operator. Section 9.1 introduces a model problem for an elliptic partial differential equation with non-smooth coefficients. Section 9.2 describes a construction for a low-rank approximation of the solution operator of the partial differential equation. Section 9.4 applies the error estimates from Chapter 6 to prove that the discrete solution operator can be approximated by an efficient $\mathcal{H}^2$-matrix.

Chapter 10 is devoted completely to applications: Section 10.1 presents a simple boundary integral equation related to Poisson's equation on a bounded or unbounded domain; in Section 10.2 one can read about the approximation of the operator mapping Dirichlet to Neumann boundary values of Poisson's equation. Section 10.3 demonstrates that the approximate arithmetic operations for $\mathcal{H}^2$-matrices can be used to construct efficient preconditioners for boundary integral equations. Section 10.4 shows that the presented techniques also work for non-academic examples, and, finally, the last subsection is devoted to the construction of solution operators for elliptic partial differential equations.

The book has a rich bibliography of 108 references (about 20 of them are the author's papers) and it is concluded by a comprehensive algorithms index and subject index.

The monograph under review is without any doubt a very carefully prepared one, and researchers interested in hierarchical matrices (especially $\mathcal{H}^2$-matrices) have to get in touch with this book.

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