# BOOK REVIEWS

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*Basic noncommutative geometry, 2nd edition*, by Masoud Khalkhali, EMS Series of Lectures in Mathematics, European Mathematical Society (EMS), Zürich, 2013, xviii+239 pp., ISBN 978-3-03719-128-6; online 978-3-03719-628-1, €38.00

Looking back at the history of mathematics, among major developments one can identify *two trends* and *one big idea* that are related to the subject of the book under review. On the one hand, mathematicians have been generalizing the notion of *space* to an ever more increasing complexity. At the same time the concept of *number* or *algebraic system* saw constant generalization. An important idea here is that these two trends go almost always hand in hand. Geometric intuition is helped by algebraic formalism and vice versa. Moreover, mathematicians believe, almost as dogma, that a statement in any kind of algebra should have a corresponding statement in geometry and vice versa. This derives from the fact that the information about a space can be encoded in terms of algebras of functions, or more exotic algebraic objects, on that space.

Thus, starting from Euclid's one-, two-, and three-dimensional spaces and simple geometric figures like lines, triangles, circles, and spheres in them, we have learned how to think and work with higher-dimensional Euclidean spaces, and with curves and surfaces and the higher-dimensional analogues living in them. This has led to the birth of projective and algebraic geometry, which dominated nineteenth-century mathematics and onward. Next came the ground-breaking and liberating work of Riemann, where one encounters a new notion of space, which he calls a *mannigfaltigkeit*, which exists on its own without regard to an ambient Euclidean space. It should be stressed that Riemann's notion of *mannigfaltigkeit* is more general than what in modern mathematics is called a *manifold.* He clearly considers discrete sets, whose only distinctive property is their cardinality, as well as infinite-dimensional function spaces to be on the same footing and as examples of his *mannigfaltigkeits.* Riemann in fact came very close to the notion of a *set.*[1] By the work of Cantor, Hausdorff, and others, set theory was understood as the ultimate building ground upon which all kinds of spaces can be erected. Thanks to set theory and starting with the notion of *topological space*, new notions, such as manifolds, varieties, schemes, stacks, etc., could be rigorously introduced and studied. *Noncommutative*

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[1]It is interesting to note that that Cantor in his first papers on set theory uses the term *mannigfaltigkeit* for a set. Only later he coined the term *menge* as the German word for set.

*spaces*, exotic as they seem to be, are the next step in this saga of expansion of our geometric intuition.

Parallel to extensions of the notion of space, mathematics has witnessed the creation of more and more complex algebraic systems that we simply refer to here as *algebra*. Thus starting with natural numbers, integers and rational numbers were created, culminating in the first rigorous definitions of real numbers by Dedekind and Cantor in the 19th century. The creation of quaternions by Hamilton marks the beginning of a new era in algebra. For the first time quantities were considered as members of an algebraic system where multiplication was not commutative: $ab$ is not the same as $ba$. This has eventually led to the creation of *hypercomplex numbers*, or *associative algebras* in more modern terminology. In particular, *matrix algebras* and their infinite-dimensional analogues, $C^*$ and von Neumann algebras of operators on Hilbert space, were defined and intensive studies of them began in the twentieth century.

The marriage or unification of (commutative) algebra and (classical) geometry has been going on almost from the beginning of mathematical thought. Descartes took a decisive step in this regard by introducing coordinates in the plane and in space and thus turning questions of geometry into problems of algebra and vice versa. Now we can think of an expression like $x^2 + y^2 = 1$ as the equation of a circle. A modern and much more precise formulation of Descartes's idea is *Hilbert's Nullstellensatz* that can be formulated as an equivalence between the category of affine schemes over an algebraically closed field and the opposite of the category of reduced commutative and finitely generated algebras over that field. In functional analysis, the celebrated *theorem of Gelfand and Naimark* gives a similar equivalence between the category of compact Hausdorff spaces and the opposite of the category of unital commutative $C^*$-algebras. Noncommutative geometry builds on, and vastly extends, this fundamental duality between classical geometry and commutative algebras. Thus one can think of the category of not necessarily commutative $C^*$-algebras as the dual of an otherwise undefined category of *noncommutative locally compact spaces*. What makes this a successful proposal is a rich supply of examples and also the possibility of extending many of the topological and geometric invariants and tools of geometric analysis to this new class of "spaces".

*Noncommutative geometry*, in the sense that is studied in this book, is the brain child of the mathematician Alain Connes, who has been its main architect and visionary in the past 35 years. It builds primarily on the idea that regarding particular classes of noncommutative algebras as algebras of coordinates on a fictitious noncommutative space can be very useful.

Why can noncommutative spaces be a useful idea? The inadequacy of classical spaces is clear when we we deal with highly singular so-called *bad quotients*. Spaces, such as the quotient of a manifold by the ergodic action of a group or the space of leaves of a foliation, are typically ill behaved. Another example of being an ill-behaved space is, for instance, that they may fail to be even Hausdorff or to have enough open sets, let alone being a reasonably smooth space. The unitary dual of a discrete group, except when the group is abelian or almost abelian, is another example of an ill-behaved space.

In Chapter 1, some of the most fundamental *algebra-geometry correspondences*, or duality theorems, that form the backbone of the subject are treated with much detail. The Gelfand–Naimark theorem, Hilbert's *Nullstellensatz*, Riemann surfaces and their function fields, affine schemes, the Serre–Swan theorem, Hopf algebras,

Table 1

| **commutative** | **noncommutative** |
|---|---|
| measure space | von Neumann algebra |
| locally compact space | $C^*$-algebra |
| vector bundle | finite projective module |
| complex variable | operator on a Hilbert space |
| real variable | self-adjoint operator |
| infinitesimal | compact operator |
| range of a function | spectrum of an operator |
| $K$-theory | $K$-theory |
| vector field | derivation |
| integral | trace |
| closed de Rham current | cyclic cocycle |
| de Rham complex | Hochschild homology |
| de Rham cohomology | cyclic homology |
| Chern character | Connes–Chern character |
| Chern–Weil theory | noncommutative Chern–Weil thoery |
| elliptic operator | $K$-cycle |
| spin$^c$ Riemannian manifold | spectral triple |
| index theorem | local index formula |
| group, Lie algebra | Hopf algebra, quantum group |
| symmetry | action of Hopf algebra |

and quantum groups are among the topics that are covered in this chapter. They lead to ideas of noncommutative locally compact spaces, noncommutative affine varieties, and noncommutative vector bundles. Several examples of noncommutative spaces, most notably noncommutative tori, group $C^*$-algebras, and quantum groups are treated with many details. The last section of this first chapter is a self-contained introduction to Hopf algebras and quantum groups and the idea of symmetry in noncommutative geometry. There is an open and growing dictionary on the subject that builds on these correspondences and gives the noncommutative analogues of many notions of geometry and topology. One can find one such a dictionary already in the introduction of this book (see Table 1).

In Chapter 2 the author discusses formation of noncommutative quotients via groupoids and groupoid algebras and gives an excellent array of examples. It is shown how one can replace a bad classical quotient by a noncommutative algebra which behaves nicely, provided one has the right tools. This is one of the most universal and widely used methods for constructing noncommutative spaces. Another important concept in this chapter is the idea of Morita equivalence of algebras, both at purely algebraic and $C^*$-algebraic levels. Among other things, Morita equivalence clarifies the relation between noncommutative quotients and classical quotients.

Cyclic cohomology, discovered by Alain Connes in 1981, is at the heart of noncommutative geometry. It should be seen as a noncommutative analogue of de Rham homology of currents. In fact, Chapter 3 starts by quoting Alain Connes's summary of his 1981 Oberwolfach talk, where he unveiled this notion for the first time. It is fascinating to see how the need to extend the index theorem to foliation algebras led Connes to an algebraic theory—one that is much appreciated by workers in algebraic $K$-theory and homotopy theory as well. Together with $K$-theory, $K$-homology, and $KK$-theory in general, (periodic) cyclic cohomology gives a homotopy invariant of noncommutative algebras which is a receptacle for a Chern character map. Cyclic (co)homology, its relation with Hochschild (co)homology through Connes's long sequence and spectral sequence, and its relation with de Rham (co)homology are treated at length in this chapter. Three different definitions of cyclic cohomology are given, each shedding light on a different aspect of the theory. Continuous versions of cyclic and Hochschild cohomology for topological algebras is developed in this section as well. This plays an important role in applications.

Chapter 4 is the culmination of the theory developed in the book. It aims at giving a proof of Connes's noncommutative index theorem for finitely summable Fredholm modules:

$$\begin{array}{ccccc} \mathfrak{K}^i(A) & \times & K_i(A) & \xrightarrow{index} & \mathbb{Z} \\ \downarrow \mathrm{Ch}^i & & \downarrow \mathrm{Ch}_i & & \downarrow \\ HP^i(A) & \times & HP_i(A) & \longrightarrow & \mathbb{C} \end{array}$$

The diagram, which appears on the book cover, in fact should be seen as a way of formulating an index theorem in a noncommutative setting in general, and it is a prototype of such results. It should be understood as equality of an analytic index with a topological index.

Thus the author defines Connes–Chern character maps for both $K$-theory and $K$-homology. For $K$-theory, it is the noncommutative analogue of the classical Chern character map from $K$-theory to de Rham cohomology. It can also be described as a pairing between $K$-theory and cyclic cohomology. Fredholm modules, as cycles for $K$-homology, are introduced next and, for finitely summable Fredholm modules, their Connes–Chern character with values in cyclic cohomology is introduced. These pairings are then used to prove an index formula of Connes relating the analytic Fredholm index of a finitely summable Fredholm module to its topological index. This is an example of an index formula in noncommutative geometry. The very last section of this chapter summarizes many ideas of the book into one commutative diagram, which is the above-mentioned index formula. A nice application of this index theorem is an integrality result which was used by Connes to show that there are no nontrivial projections in the (reduced) group $C*$-algebra of a free group with two generators (the Kadison–Kaplansky conjecture). No purely analytic proof of this fact is known. Another fascinating application of this result is the integrality of quantum Hall conductance. The conductance can be expressed as the pairing between a cyclic 2-cocycle and a $K$-theory class of a noncommutative algebra. The integrality follows by showing that the cyclic cocycle is the Chern character of a Fredholm module over the algebra.

There are also appendices covering basic material on $C^*$-algebras, compact and Fredholm operators, projective modules, and category theory language.

Khalkhali's book is an excellent introduction for beginners to noncommutative geometry and some of its applications. It should be very useful for preparing the reader for more advanced topics like *Baum–Connes conjectures* and the *local index formula*. There are many examples and exercises given in almost every section of the book. This makes the book ideal as a textbook or for self study. As such it should be valuable for students of pure mathematics and theoretical physics. In the new edition, two new sections are added that cover more recent developments in the subject. A study of curvature invariants and Gauss–Bonnet type theorems using heat equation techniques for a noncommutatve 2-torus is a very interesting new development in the subject that is briefly discussed. Also added is a succinct introduction to Hopf cyclic cohomology and its applications.

Herman Weyl once famously said, "In these days the angel of topology and the devil of abstract algebra fight for the soul of every individual discipline of mathematics." In noncommutative geometry, it seems we have a perfect collaboration of angels of algebra, geometry, and physics to lift the two subjects to higher heights!

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