Bull. London Math. Soc. 43 (2011) 206–208

BOOK REVIEW

Basic noncommutative geometry
(EMS Series of Lectures in Mathematics)

By Masoud Khalkhali: 223 pp., €36, ISBN 978-3-0371-9061-6
(European Mathematical Society, Zürich, Switzerland, 2009).

doi:10.1112/blms/bdq100
Published online 8 November 2010

This is a book on non-commutative geometry, a subject in which the reviewer is not an expert, but this book is an introduction, and so perhaps I am the target audience. Regardless, I certainly enjoyed the read.

A rather common theme in modern mathematics is not to study objects directly, but rather to study maps between objects; commonly, in fact, we associate some sort of algebra with an object. Perhaps non-commutative geometry starts with the observation that a suitably chosen algebra can encode a great deal of the structure of the original object.

This book is firmly in the Connes school, and is strongly influenced by functional analysis and differential geometry. The canonical example here is the fact that there is a functor between the category of compact Hausdorff spaces with continuous maps and the category of commutative unital C*-algebras with $*$-homomorphisms. To a compact space X we associate the algebra $C(X)$ of complex-valued continuous functions on X; this is a unital commutative C*-algebra, and every such algebra arises in this way. In fact, these categories are opposite; and so we might, perhaps a touch naïvely, say that a unital non-commutative C*-algebra represents the 'algebra of functions' on a 'non-commutative' compact space. The first chapter explores further links between spaces and algebras, most examples coming from algebraic and differential geometry, and from the theory of Hopf algebras (loosely, 'quantum groups', although links with operator algebraic approaches, for example the work of Woronowicz, are rather glossed over).

One of the principle motivations behind non-commutative geometry is that algebras can 'remember' more information than spaces. This occurs when looking at equivalence relations. For example, if we consider a rational rotation of the circle $\mathbb{T}$, then each orbit is finite, and the quotient space is easily understood. By contrast, the orbits of an *irrational* rotation are all dense, and the resulting quotient space is uncountable, but has the indiscrete topology. At the level of algebras, we can instead start with the algebra $C(\mathbb{T})$, and a rotation of $\mathbb{T}$ induces an action of $\mathbb{Z}$ on $C(\mathbb{T})$. We can thus form the crossed product $C(\mathbb{T}) \rtimes \mathbb{Z}$. Even in the rational rotation case, this algebra remembers more information about the action than the classical quotient algebra does, as the latter is always $C(\mathbb{T}/\mathbb{Z}) \cong C(\mathbb{T})$. Conversely, if a (say discrete) group G acts freely and properly on compact space X, then the algebras $C(X) \rtimes G$ and $C(X/G)$ are related: they are not isomorphic, but instead satisfy the weaker property of being *Morita equivalent*, a nice example of an algebraic idea capturing something geometrical in nature. Chapter 2 of the book explores these ideas in the more general setting of groupoid algebras.

Chapter 3 switches gears, and explores (co)homology theories. The aim is to develop cyclic cohomology, in the sense of Connes, Tsygan, Loday and Quillen. First we explore Hochschild cohomology, mainly with coefficients in the dual space A^* of an algebra A. Recall that this is a complex involving the spaces $\hom(A^{\otimes(n+1)}, \mathbb{C})$ together with a differential b satisfying $b^2 = 0$ (the definition of b is complicated, but presumably familiar to many readers).

The cohomology then measures the difference between the image of b and the kernel of b. Many examples are given; a principle motivation is to convince the reader that the mth continuous Hochschild cohomology of $C^\infty(M)$ with values in its dual is isomorphic to the space of m-currents on M; similarly homology is isomorphic to forms on M. This allows us to view the Hochschild (co)homology of a non-commutative algebra as, in some sense, being equivalent to forms and currents on a non-commutative space. Then deformation quantizations are explored, and various other examples explored. Note that here it makes sense to work with topological algebras rather than operator algebras: if M is a smooth compact manifold, then $C(M)$ only sees the topology of M, and not the manifold structure, whereas $C^\infty(M)$ does capture (at least some of) the geometric structure of M.

In the second half of the chapter cyclic cohomology is introduced. This is first defined by looking at a subcomplex of the Hochschild complex: the basic idea is to look at the subspace of $\hom(A^{\otimes(n+1)}, \mathbb{C})$ that is (graded) invariant under the cycle (so $f(a_0, \ldots, a_n) = (-1)^n f(a_n, a_0, \ldots, a_{n-1})$). This is compatible with the Hochschild differential b, and so we get a cohomology, say $HC^*(A)$. Links are made to Stokes's theorem when studying $C^\infty(M)$. This can be generalized in a surprising way by starting with an algebra A and then defining ΩA to be just the universal differential graded algebra generated by A. The cyclic cocycles on A are just closed graded traces of ΩA. This gives useful geometric intuition to the study of cyclic cohomology. Furthermore, this leads to a simple definition of the periodicity operator S which maps the nth cyclic cohomology to the $(n+2)$th cyclic cohomology. Then the periodic cyclic cohomology $HP^k(A)$ can be defined as the direct limit of $HC^{2n+k}(A)$ for $k = 0, 1$. After this, Connes's long exact sequence and spectral sequence are studied: these are powerful computational tools which (for the long exact sequence) make links with Hochschild cohomology. Then a homological algebra approach is taken by looking at cyclic modules. At the end various calculations are given for $C^\infty(M)$ and for group algebras. For example, $HP^k(C^\infty(M))$ is the direct sum of the de Rham cohomology $H^{dR}_{2i+k}(M)$, for $k = 0, 1$.

The final chapter is on the 'Connes–Chern character'. The idea is to relate K-theory to cyclic homology, or equivalently to define pairings between cyclic cohomology and K-theory. The chapter recalls the basics of K-theory, both from a projective module perspective, and also the more familiar formulation in terms of idempotents in matrix algebras. Then the pairing between K-theory and cyclic cohomology (with values in the dual of the algebra) is defined. This is compatible with the periodicity operator, and so we really get two pairings $HP^i(A) \otimes K_i(A) \to \mathbb{C}$ for $i = 0, 1$, called the Connes–Chern pairings. Again, this is explored through numerous examples. The next section explores K-homology, and here we are introduced to Fredholm modules: these are a Hilbert space H upon which our algebra A is represented, together with an operator F such that $F^2 = 1$ and the commutant of F with an operator in A is compact (at least, in the odd case – in the even case, we need to take a grading of H). If we can replace 'compact' by 'Schatten class' then our Fredholm module has a summability condition, and this yields cyclic cocycles. A worked example is given for $C^*_r(\mathbb{F}_2)$, the (reduced) group C*-algebra of the free group on two generators. Then a pairing between Fredholm modules and K-theory is defined, using the index theory of Fredholm operators. Applied to the example of $C^*_r(\mathbb{F}_2)$, this gives Connes' short proof that $C^*_r(\mathbb{F}_2)$ is 'connected': it has no non-trivial projections.

In the proof of this is a little functional calculus argument. This is picked out in the last section to explore further the pairing between cyclic cohomology and K-theory. As we saw, cyclic cohomology seems most profitably applied to topological algebras like $C^\infty(M)$. By contrast, K-theory is a theory of C*-algebras, and so we would want to work with $C(M)$ not $C^\infty(M)$ in this example. However, the inclusion $C^\infty(M) \subseteq C(M)$ induces an isomorphism of K-theory. This is true for a wide class of inclusions $\mathcal{A} \subseteq A$, when $\mathcal{A}$ is stable under holomorphic functional calculus: that is, if we apply the holomorphic functional calculus for the Banach algebra A to an element of $\mathcal{A}$, we land in $\mathcal{A}$ and not the larger algebra A. The link with previous ideas is that if (H, F) is a Fredholm module over A, then if $\mathcal{A}$ consists of those operators satisfying our summability condition, $\mathcal{A}$ is stable under holomorphic functional calculus. A worked example for the Toeplitz algebra is given.

The book ends with an explanation of how an algebraic 'index theory' defined by (periodic) cyclic homology and cohomology is the same as the analytic index theory coming from Fredholm modules and K-theory. Indeed, this explains the commutative diagram on the cover of the book! Finally, some hints are given that a better theory can be obtained by weakening the notion of a Fredholm module to give a spectral triple (our operator F is allowed to be unbounded, subject to some further conditions).

As suggested, this is not a monograph, and proofs are often not given. However, the text is full of examples, enough detail is always given to convince the reader that unproved results are reasonable, and often hints of the proof are given. References are always given.

The reviewer's background is in functional analysis and abstract harmonic analysis. I had little problem following the algebra and differential geometry in the book, and there are some useful appendices that quickly sketch some of the operator algebra theory needed, although this would be rough going if the reader had not seen at least some of it before. At 200 pages, this is not a short book, but it is a real joy to read, and the reviewer sped through it. There are a couple of typos, but as far as I could see, nothing serious. The index is very useful, and there are plenty of interesting exercises.

My one criticism might be that, even by the end of the book, it is still not really clear in what direction non-commutative geometry is travelling. What are the big open problems? However, this is a technical subject, and the book is sprinkled with nice applications. The bibliography is large, and, while not perfect, gives plenty of pointers for further reading. In the end, this is only an 'introduction', and in that sense I think it succeeds admirably.

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