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BOOK REVIEW

# *Dynamical Systems and Processes*

(IRMA Lectures in Mathematics and Theoretical Physics 14)

*By* Michel Weber: 773 pp., €98.00, ISBN 978-3-03719-046-3
(European Mathematical Society, Zürich, Switzerland, 2009).


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Mathematicians have always loved iteration. If we have a mapping $T : X \to X$ and consider the effect of $T$, $T^2$, ..., will the successive iterations converge? If not, perhaps the average will converge (in some sense). Thus, we have Weyl's theorem that, working on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with $f : \mathbb{T} \to \mathbb{R}$ continuous,

$$(n+1)^{-1} \sum_{j=0}^{n} f(n\alpha) \longrightarrow \int_{\mathbb{T}} f(x)\, dx,$$

whenever $\alpha$ is irrational.

If we equip $X$ with a probability measure $\mu$, then we have one of the great theorems of the twentieth century, Birkhoff's pointwise ergodic theorem, which states that if $T$ is a measure preserving bijection and $f : X \to \mathbb{R}$ is integrable, then

$$S_n(f, x) = (n+1)^{-1} \sum_{j=0}^{n} f(T^n x)$$

converges almost everywhere as $n \to \infty$. Under a wide variety of conditions on $T$, this result strengthens to give

$$(n+1)^{-1} \sum_{j=0}^{n} f(T^n x) \longrightarrow \int_{X} f(x)\, dx.$$

Of course, there are other possible modes of convergence. Von Neumann's ergodic theorem tells us that, if $f$ is square summable, then $S_n(f)$ converges in $L^2$. Working in $L^2$ gives us the full power of Hilbert space theory. The first of the four parts of the book deals with developments of von Neumann's theorem. The second deals with developments of Birkhoff's theorem.

The third part approaches matters from the point of view of stochastic processes, leading to a discussion of random Fourier series which left me feeling like a native of Poughkeepsie visiting New York for the first time. The final part discusses some special questions.

It is not surprising that this mixture of the particular (for example, the dependence on the irrationality of $a$ in Weyl's theorem) and the staggeringly general has attracted some of the most gifted mathematicians of our time. This book can be considered as a meditation on these themes. Although approaching the encyclopaedic in length, it is rather personal in tone.

As an example, consider a sequence of independent identically distributed random variables $X_0$, $X_1$, $X_2$, .... A simple application of Birkhoff's theorem yields the strong law of large numbers: if $\mathbb{E}|X_0| < \infty$, then, with probability 1,

$$(n+1)^{-1} \sum_{j=0}^{n} X_j \longrightarrow \mathbb{E}X_0.$$

However, we know that if $X_0$ is well behaved (with $\mathbb{E}|X_0|^2 = \sigma^2 < \infty$ and $\mu = \mathbb{E}X_0$), then the central limit theorem tells us that

$$\Pr\left(\frac{\sum_{j=0}^{n}(X_j - \mu)}{\sigma(n+1)^{1/2}} \in [a,b]\right) \longrightarrow \frac{1}{2\pi}\int_a^b \exp(-x^2/2)\,dx.$$

If, like me, you have never even thought of the question 'what is the Birkhoffian analogue of the central limit theorem?', then you will be delighted to find both the question and its answer.

In 700 pages, the author finds plenty of time to discuss both the general and the particular, but I suspect that his heart belongs to the particular, as in the study of the convergence of series of the sums of the form $\sum_{j=0}^{n} c_k f(n_k x)$ and the chapter on conditions for the almost sure convergence of Riemann sums of Lebesgue integrable functions.

It is inevitable that, in a subject which mathematicians like Bourgain and Talagrand have found worthy of their talents, strikingly elegant results are often matched with strikingly difficult proofs. However, the author has provided a clear and enthusiastic account which should find its way into the hands of ambitious graduate students in analysis and probability.

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