

# Introduction to Teichmüller theory, old and new, II

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## Contents

1	The metric and the analytic theory . . . . .	2
1.1	Weil–Petersson geometry . . . . .	2
1.2	The quasiconformal theory . . . . .	4
1.3	Holomorphic families . . . . .	6
1.4	Uniformization . . . . .	9
1.5	Combinatorial classes . . . . .	9
1.6	Differential forms . . . . .	12
2	The group theory . . . . .	13
2.1	Quasi-homomorphisms . . . . .	13
2.2	Lefschetz fibrations . . . . .	15
2.3	Measure-equivalence . . . . .	17
2.4	Affine groups . . . . .	20
2.5	Braid groups . . . . .	22
3	Representation spaces and geometric structures . . . . .	25
3.1	Complex projective structures . . . . .	26
3.2	Circle packings . . . . .	29
3.3	Lorentzian geometry . . . . .	31
3.4	Fricke–Klein coordinates . . . . .	34
3.5	Diagrammatic approach . . . . .	34
4	The Grothendieck–Teichmüller theory . . . . .	35
4.1	The reconstruction principle . . . . .	38
4.2	Dessins d’enfants . . . . .	39
4.3	The solenoid . . . . .	41

This introduction can be considered as a sequel to the introduction that I wrote for Volume I of the Handbook, and I shall limit myself here to a general presentation of the material covered in the present volume. The exposition will follow the four-parts division of the volume, and for each part, its division in chapters.

Beyond the information given on the content of this volume, I hope that the reader of this introduction will get (if he does not have it yet) an idea of the richness of the subject of Teichmüller theory.

All the surfaces considered in this introduction are orientable, unless otherwise stated. I have tried to give some necessary definitions to make the introduction as much self-contained as possible.

## 1 The metric and the analytic theory

Part A of this volume, on the metric and analytic theory of Teichmüller space, contains chapters on Weil–Petersson geometry, on biholomorphic maps between finite or infinite-dimensional Teichmüller spaces, on the theory of holomorphic families of Riemann surfaces, on uniformization of algebraic surfaces, on combinatorial classes in moduli space and on canonical differential forms on that space representing cohomology classes.

### 1.1 Weil–Petersson geometry

Chapter 1 by Scott Wolpert is a review of some recent work on the Weil–Petersson metric of  $\mathcal{T}_{g,n}$ , the Teichmüller space of a surface of genus  $g \geq 0$  with  $n \geq 0$  punctures, with negative Euler characteristic. Let us start by recalling some basic facts about this metric.

It is well known that the cotangent space to  $\mathcal{T}_{g,n}$  at a point represented by a Riemann surface  $S$  can be identified with the space  $Q(S)$  of holomorphic quadratic differentials on  $S$  that have at most simple poles at the punctures. The Weil–Petersson cometric on that cotangent space is given by the Hermitian product  $\int_S \phi(z)\bar{\psi}(z)\rho^{-2}(z)|dz|^2$ , for  $\phi$  and  $\psi$  in  $Q(S)$ , where  $\rho$  is the density form of the length element  $\rho(z)|dz|$  of the unique complete hyperbolic metric that uniformizes the Riemann surface  $S$ .<sup>1</sup>

The Weil–Petersson metric on  $\mathcal{T}_{g,n}$  is Kähler, geodesically convex and with negative and unbounded sectional curvature (its supremum is zero, and its infimum is  $-\infty$ ). Its Ricci curvature is bounded from above by a negative constant. This metric is not complete, and a geodesic of bounded length can be obtained by making the hyperbolic length of a closed geodesic on the surface tend to zero. The last fact explains intuitively why the completion of the Weil–Petersson metric gives rise to the *augmented* Teichmüller space  $\bar{\mathcal{T}}_{g,n}$ , whose elements are equivalence classes of marked stable Riemann surfaces, that is, marked Riemann surfaces with nodes, with the property that each connected component of the complement of the nodes is a surface with cusps which has negative Euler characteristic. The space  $\bar{\mathcal{T}}_{g,n}$  is a stratified space which is not locally compact and which is a partial compactification of  $\mathcal{T}_{g,n}$ . The action of the mapping class group on  $\mathcal{T}_{g,n}$  extends to an action on  $\bar{\mathcal{T}}_{g,n}$ , and the quotient of  $\bar{\mathcal{T}}_{g,n}$  by

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<sup>1</sup>The name Weil–Petersson has been given to this metric because it was André Weil who first noticed that this product, called the Petersson product and originally introduced by Hans Petersson on the space of modular forms, gives a metric on Teichmüller space.

this action is a compact orbifold, known as the Deligne–Mumford stable curve compactification of moduli space. In 1976, H. Masur obtained a beautiful result stating that the Weil–Petersson metric on Teichmüller space extends to a complete metric on the augmented Teichmüller space  $\overline{\mathcal{T}}_{g,n}$ . This result is one of the starting points for a topological approach to the Weil–Petersson metric.

Our knowledge of the Weil–Petersson geometry underwent a profound transformation at the beginning of 1980s, thanks to the work of Scott Wolpert, who obtained a series of particularly elegant results on the Weil–Petersson metric and on its associated symplectic form. New important results on the subject, from various points of view, were obtained in the last few years by several authors, including Wolpert, Yamada, Huang, Liu, Sun, Yau, McMullen, Mirzakhani, Brock, Margalit, Daskalopoulos and Wentworth (there are others). The recent work on Weil–Petersson geometry includes the study of the CAT(0) geometry of augmented Teichmüller space, that is, the study of its nonpositive curvature geometry in the sense of Cartan–Alexandrov–Toponogov (following a terminology introduced by Gromov). We recall that the definition of a CAT(0) metric space is based on the comparison of distances between points on the edges of arbitrary triangles in that metric space with distances between corresponding points on “comparison triangles” in the Euclidean plane. It is known that augmented Teichmüller space, equipped with the extension of the Weil–Petersson metric, is a complete CAT(0) metric space (a result due to Yamada). The Weil–Petersson isometry group action extends continuously to an action on augmented space. The Weil–Petersson isometry group coincides with the extended mapping class group of the surface except for some special surfaces (a result of Masur & Wolf, completed to some left-out special cases by Brock & Margalit, which parallels a famous result by Royden for the Teichmüller metric, completed by Earle & Kra). An analysis of the action of the mapping class group in the spirit of Thurston’s classification of mapping classes, showing in particular the existence of invariant Weil–Petersson geodesics for pseudo-Anosov mapping classes, has been carried out by Daskalopoulos & Wentworth. Brock established that (augmented) Teichmüller space equipped with the Weil–Petersson metric is quasi-isometric to the pants graph of the surface.

In Chapter 1 of this volume, Wolpert makes a review of the recent results on the metric aspect (as opposed to the analytical aspect) of the Weil–Petersson metric. He reports on a parametrization of augmented Teichmüller space using Fenchel–Nielsen coordinates and on a comparison between the Weil–Petersson metric and the Teichmüller metric in the thin part of Teichmüller space, using these coordinates. He gives formulae for the Hessian and for the gradient of the hyperbolic geodesic length functions and for the behaviour of these functions near degenerate hyperbolic surfaces. He also gives formulae for the Weil–Petersson symplectic form in terms of geodesic length functions. Weil–Petersson convexity and curvature are also reviewed. The chapter also contains a section on Alexandrov angles, in relation with Alexandrov tangent cones at points of the augmented Teichmüller space. Wolpert gives estimates on the exponential map, with applications to the first variation formula for the distance and to the length-minimizing paths connecting two given points and intersecting a

prescribed stratum. He displays a table comparing the known metric properties of the Teichmüller space of a surface of negative Euler characteristic with corresponding properties of the hyperbolic plane, which, as is well known, is the Teichmüller space of the torus.

## 1.2 The quasiconformal theory

In Chapter 2, Alastair Fletcher and Vladimir Markovic study analytic properties of finite-dimensional as well as infinite-dimensional Teichmüller spaces. They review some classical properties and they present some recent results, in particular concerning biholomorphic maps between Teichmüller spaces.

We recall that a Riemann surface is said to be of *finite topological type* if its fundamental group is finitely generated. It is said of *finite analytical type* if it is obtained (as a complex space) from a closed Riemann surface by removing a finite set of points. The Teichmüller space  $\mathcal{T}(S)$  of a Riemann surface  $S$  is a Banach manifold which is finite-dimensional if and only if  $S$  is of finite analytical type. (Note that  $\mathcal{T}(S)$  can be infinite-dimensional even if  $S$  has finite topological type.) A surface with border has an *ideal boundary*, which is the union of its *ideal boundary curves*, and the Teichmüller space of a surface with nonempty border is infinite-dimensional. The most important surface with border is certainly the unit disk  $\mathbb{D} \subset \mathbb{C}$ , and its Teichmüller space is called *universal Teichmüller space*. This space contains all Teichmüller spaces of Riemann surfaces, as we shall recall below.

In this chapter,  $S$  is a surface of finite or infinite type.

The Teichmüller space  $\mathcal{T}(S)$  of a Riemann surface  $S$  is defined as a space of equivalence classes of marked Riemann surfaces  $(S', f)$ , with the marking  $f$  being a quasiconformal homeomorphism between the base surface  $S$  and a Riemann surface  $S'$ . We recall that for infinite-dimensional Teichmüller spaces, the choice of a base Riemann surface is an essential part of the definition, since homeomorphic Riemann surfaces are not necessarily quasiconformally equivalent. Teichmüller space can also be defined as a space of equivalence classes of Beltrami differentials on a given base Riemann surface. The relation between the two definitions stems from the fact that a quasiconformal mapping from a Riemann surface  $S$  to another Riemann surface is the solution of an equation of the form  $f_{\bar{z}} = \mu f_z$  (called a Beltrami equation), with  $\mu$  a Beltrami differential on  $S$ .

Fletcher and Markovic also deal with universal Teichmüller space. This is a space of equivalence classes of normalized quasiconformal homeomorphisms of the unit disk  $\mathbb{D}$ . It is well known that quasiconformal maps of  $\mathbb{D}$  extend to the boundary  $\partial\mathbb{D}$  of  $\mathbb{D}$ . Such quasiconformal maps are normalized so that their extension to the boundary fixes the points  $1$ ,  $-1$  and  $i$ , and two quasiconformal self-maps of the disk are considered to be equivalent if they induce the same map on  $\partial\mathbb{D}$ . Like the other Teichmüller spaces, universal Teichmüller space can also be defined as a space of equivalence classes of Beltrami differentials. By lifting quasiconformal homeomorphisms or Beltrami differentials from a surface to the universal cover, the Teichmüller space of any surface

of hyperbolic type embeds in the universal Teichmüller space, and it is in this sense that the universal Teichmüller space is called “universal”.

The complex Banach structure of each Teichmüller space  $\mathcal{T}(S)$  can be obtained from the so-called Bers embedding of  $\mathcal{T}(S)$  into the Banach space  $Q(S)$  of holomorphic quadratic differentials on the base surface  $S$ . In the case where  $S$  is of finite analytical type, this embedding provides a natural identification between the cotangent space at a point of  $\mathcal{T}(S)$  and a Banach space  $Q$  of integrable holomorphic quadratic differentials, and the two spaces are finite-dimensional. In the general case, the spaces considered are not necessarily finite-dimensional, and the cotangent space at a point of  $\mathcal{T}(S)$  is the *predual* of the Banach space  $Q$ , that is, a space whose dual is  $Q$ . The predual of  $Q$  is called the *Bergman space* of  $S$ . This distinction, which is pointed out by Fletcher and Markovic, is an important feature of the theory of infinite-dimensional Teichmüller spaces.

It is well known that the complex-analytic theory of finite-dimensional Teichmüller spaces can be developed using more elementary methods than those that involve the Bers embedding. For instance, for surfaces of finite analytical type, Ahlfors defined the complex structure of Teichmüller space using period matrices obtained by integrating systems of independent holomorphic one-forms over a basis of the homology of the surface. The complex analytic structure on Teichmüller space is then the one that makes the period matrices vary holomorphically. The description of the complex structure in the infinite-dimensional case requires more elaborate techniques.

Along the same line, we note some phenomena that occur in infinite-dimensional Teichmüller theory and not in the finite-dimensional one. There is a “mapping class group action” on infinite-dimensional Teichmüller spaces, but, unlike the finite dimensional case, this action is not always discrete. (Here, discreteness means that the orbit of any point under the group action is discrete.) Katsuhiko Matsuzaki studied limit sets and domains of discontinuity for such actions, in the infinite-dimensional case. From the metric-theoretic point of view, Zhong Li and Harumi Tanigawa proved that in each infinite-dimensional Teichmüller space, there are pairs of points that can be connected by infinitely many distinct geodesic segments (for the Teichmüller metric). This contrasts with the finite-dimensional case where the geodesic segment connecting two given points is unique. Li proved non-uniqueness of geodesic segments connecting two points in the universal Teichmüller space, and he showed that there are closed geodesics in any infinite-dimensional Teichmüller space. He also proved that the Teichmüller distance function, in the infinite-dimensional case, is not differentiable at some pairs of points in the complement of the diagonal, in contrast with the finite-dimensional case where, by a result of Earle, the Teichmüller distance function is continuously differentiable outside the diagonal.<sup>2</sup>

The mention of these differences between the finite- and infinite-dimensional cases will certainly give more importance to the results on isometries and biholomorphic

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<sup>2</sup>The study of the differentiability of the Teichmüller distance function was initiated by Royden, and it was continued by Earle. More precise results on the differentiability of this function were obtained recently by Mary Rees.

maps between infinite-dimensional Teichmüller spaces that are reported on here by Fletcher and Markovic, since these are results that hold in both the finite- and in the infinite-dimensional cases. Fletcher and Markovic study biholomorphic maps between Teichmüller spaces by examining their induced actions on cotangent spaces (and Bergman spaces). In the finite-dimensional case, the idea of studying the action on cotangent space is already contained in the early work of Royden. The action of a biholomorphic map induces a  $\mathbb{C}$ -linear isometry between Bergman spaces. Fletcher and Markovic report on a rigidity result, whose most general form is due to Markovic, and with special cases previously obtained by Earle & Kra, Lakic and Matsuzaki. The result says that any surjective  $\mathbb{C}$ -linear isometry between the Bergman spaces  $A^1(M)$  and  $A^1(N)$  of two surfaces  $M$  and  $N$  is geometric, except in the case of some elementary surfaces. Roughly speaking, the word “geometric” means here that the isometry is a composition of two naturally defined isometries between such spaces, viz. multiplication by a complex number of norm one, and an isometry induced by the action of a conformal map between the surfaces. A corollary of this result is that the biholomorphic automorphism group of the Teichmüller space of a surface of non-exceptional (finite or infinite) type can be naturally identified with the mapping class group of that surface.

As in the finite-dimensional case, this result reduces the study of biholomorphic homeomorphisms between Teichmüller spaces to the study of linear isometries between some Banach spaces. In the course of proving this result, a proof is given of the fact that the Kobayashi and the Teichmüller metrics on (finite- or infinite-dimensional) Teichmüller space agree, again generalizing a result obtained by Royden and completed by Earle & Kra for finite type Riemann surfaces.

Chapter 2 of this volume also contains the proof of a local rigidity result due to Fletcher, saying that the Bergman spaces of any two surfaces whose Teichmüller spaces are infinite-dimensional are always isomorphic, and that any two infinite-dimensional Teichmüller spaces are locally bi-Lipschitz equivalent. More precisely, Fletcher proved that the Teichmüller metric on every Teichmüller space of an infinite-type Riemann surface is locally bi-Lipschitz equivalent to the Banach space  $l^\infty$  of bounded sequences with the supremum norm.

### 1.3 Holomorphic families

A *holomorphic family of Riemann surfaces of type  $(g, n)$*  is a triple  $(M, \pi, B)$  defined as follows:

- $M$  is a 2-dimensional complex manifold (topologically, a 4-manifold);
- $B$  is a Riemann surface;
- $\pi : M \rightarrow B$  is a holomorphic map;
- for all  $t \in B$ , the fiber  $S_t = \pi^{-1}(t)$  is a Riemann surface of genus  $g$  with  $n$  punctures;
- the complex structure on  $S_t$  depends holomorphically on the parameter  $t$ .

Chapter 3, by Yoichi Imayoshi, concerns holomorphic families of Riemann surfaces. In all this chapter, it is assumed that  $2g - 2 + n > 0$ .

Why do we study holomorphic families of Riemann surfaces? One reason is that one way of investigating the complex analytic structure of Teichmüller space involves the study of holomorphic families. Another reason is that the study of degeneration of holomorphic families is related to the study of the stable curve compactification of moduli space.

To be more precise, we use the following notation: as before,  $\mathcal{T}_{g,n}$  is the Teichmüller space of a surface of type  $(g, n)$ , that is, of genus  $g$  with  $n$  punctures and  $\mathcal{M}_{g,n}$  is the corresponding moduli space. A holomorphic family  $(M, \pi, B)$  of type  $(g, n)$  gives rise to a holomorphic map  $\Phi: \tilde{B} \rightarrow \mathcal{T}_{g,n}$ , where  $\tilde{B}$  is the universal cover of  $B$ , and to a quotient holomorphic map  $\tilde{B} \rightarrow \mathcal{M}_{g,n}$  called the *moduli map* of the family. A basic combinatorial tool in the study of the holomorphic family  $(M, \pi, B)$  is its *topological monodromy*, which is a homomorphism from the fundamental group of the base surface  $B$  to the mapping class group  $\Gamma_{g,n}$  of a chosen Riemann surface  $S_{g,n}$  of type  $(g, n)$ . In Chapter 3, this homomorphism is denoted by  $\Phi_*$ , because its definition makes use of the map  $\Phi$ . It is defined through the action of the mapping class group  $\Gamma_{g,n}$  on the Teichmüller space  $\mathcal{T}_{g,n}$ .

Imayoshi reports on an important rigidity theorem stating that if  $(M_1, \pi_1, B)$  and  $(M_2, \pi_2, B)$  are locally non-trivial holomorphic families of Riemann surfaces of type  $(g, n)$  over the same base  $B$ , and if  $(\Phi_1)_* = (\Phi_2)_*$ , then  $\Phi_1 = \Phi_2$  and  $(M_1, \pi_1, B)$  is biholomorphically equivalent to  $(M_2, \pi_2, B)$ .

Imayoshi mentions an application of this rigidity theorem to the proof of the geometric Shafarevich conjecture, which states that there are only finitely many locally non-trivial and non-isomorphic holomorphic families of Riemann surfaces of fixed finite type over a Riemann surface  $B$  of finite type. This conjecture was proved by Parshin in the case where  $B$  is compact, and by Arakelov in the general case. Imayoshi and Shiga gave a variant of the proof, using the rigidity theorem stated above. Imayoshi notes that the same rigidity theorem can be used to give a proof of the geometric Mordell conjecture, which concerns the existence of holomorphic sections for holomorphic families.

A large part of the study made in Chapter 3 concerns the case where the base surface  $B$  is the unit disk in  $\mathbb{C}$  punctured at the origin. We denote by  $\Delta^*$  this punctured disk. In many ways, taking as base surface the punctured disk is sufficient for the study of the degeneration theory of holomorphic families. It may also be useful to recall here that the Deligne–Mumford stable reduction theorem for the moduli space of curves reduces the study of the stable (Deligne–Mumford) compactification of moduli space to that of holomorphic families over the punctured disk which degenerate by producing surfaces with nodes above the puncture.

In the 1960s, Kodaira began a study of holomorphic families over the punctured disk, in the special case where the fibers are surfaces of type  $(1, 0)$ . He studied in particular the behaviour of *singular fibers* of such families, that is, fibers obtained by

extending the family at the puncture. After this work, Kodaira and others considered singular fibers of more general families. This is also reported on in Chapter 3 of this Handbook.

In the case where the base surface is the punctured disk  $\Delta^*$ , the topological monodromy is a cyclic group, and it gives rise to an element of the mapping class group of a fiber, called the *topological monodromy around the origin*. This element is defined after a choice of a basepoint  $s$  in  $\Delta^*$  and after the identification of the fiber  $\pi^{-1}(s)$  above that point with a fixed marked topological surface  $S$ . The topological monodromy is then the element of the mapping class group of  $S$  that performs the gluing as one traverses the circle in  $\Delta^*$  centered at the origin and passing through  $s$ . The topological monodromy of the family is well defined up to conjugacy (the ambiguity being due to the choice of a surface among the fibers, and of its identification with a fixed marked surface).

In 1981, Imayoshi studied monodromies of holomorphic families  $(M, \pi, \Delta^*)$  in connection with the deformation theory of Riemann surfaces with nodes. In particular, he proved that the topological monodromy of a family  $(M, \pi, \Delta^*)$  is *pseudo-periodic*<sup>3</sup>, which means that this mapping class contains an orientation-preserving homeomorphism  $f$  that preserves a (possibly empty) collection  $\{C_1, \dots, C_k\}$  of disjoint homotopically nontrivial and pairwise non-homotopic simple closed curves on the surface, such that for each  $i = 1, \dots, k$ , there exists an integer  $n_i$  such that a certain power of  $f$  is the composition of  $n_i$ -th powers of Dehn twists along the  $C_i$ 's.

Imayoshi studied a map from the punctured disk to the moduli space  $\mathcal{M}_{g,n}$  of  $S$  which is canonically associated to the family  $(M, \pi, \Delta^*)$ , and he showed that this map extends holomorphically to a map from the unit disk  $\Delta$  to the Deligne–Mumford compactification  $\bar{\mathcal{M}}_{g,n}$  of  $\mathcal{M}_{g,n}$ . He showed that algebraic properties of the topological monodromy (e.g. the fact that it is of finite or infinite order) depend on whether the image of 0 by the holomorphic map  $\Delta \rightarrow \bar{\mathcal{M}}_{g,n}$  lies in  $\mathcal{M}_{g,n}$  or in  $\bar{\mathcal{M}}_{g,n} - \mathcal{M}_{g,n}$ . He also showed that the topological monodromy is of *negative type*, meaning that it can be represented by a homeomorphism  $f$  of the fiber which is either periodic, or, using the above notation, such that the Dehn twists around the  $C_i$ 's are negative Dehn twists. Chapter 3 of this volume contains a new proof of Imayoshi's 1981 result.

Y. Matsumoto & J. M. Montesinos-Amilibia and (independently) S. Takamura proved recently a converse to Imayoshi's result. More precisely, starting with any pseudo-periodic self-map of negative type of a Riemann surface  $S_{g,n}$  satisfying  $2g - 2 + n > 0$ , they constructed a holomorphic family of Riemann surfaces over the punctured disk whose monodromy is the given map up to conjugacy. Matsumoto and Montesinos-Amilibia showed that the ambient topological type of the singular fiber is determined by the monodromy.

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<sup>3</sup>Such a mapping class is of elliptic type or of parabolic type in the Bers terminology of the Thurston classification of mapping classes.



## 1.4 Uniformization

Chapter 4, by Robert Silhol, concerns the problem of uniformization of Riemann surfaces defined by algebraic equations.

By the classical Poincaré–Koebe uniformization theorem, one can associate to any compact Riemann surface  $S$  of negative Euler characteristic a Fuchsian group, that is, a discrete subgroup  $\Gamma$  of  $\mathrm{PSL}(2, \mathbb{R})$  acting on the hyperbolic plane  $\mathbb{H}^2$ , such that  $S$  is conformally equivalent to the hyperbolic manifold  $\mathbb{H}^2/\Gamma$ . All the known proofs of the uniformization theorem are rather involved, and it is not an easy matter to explicitly exhibit the hyperbolic structure  $\mathbb{H}^2/\Gamma$  that uniformizes a given Riemann surface  $S$ . Silhol discusses this problem for the case where the Riemann surface  $S$  is given explicitly as an algebraic curve over  $\mathbb{C}$ , that is, as the zero set of a two-variable polynomial with coefficients in  $\mathbb{C}$ . We recall that by a result of Riemann, any compact Riemann surface can be defined as an algebraic curve. We note in passing that the question of what is the “best” field of coefficients for a polynomial defining a given Riemann surface can be dealt with in the setting of Grothendieck’s theory of dessins d’enfants, which is treated in another chapter of this volume. It is also worth noting that defining Riemann surfaces by algebraic equations does not necessarily reveal all the aspects of the complex structure of that surface. For instance, the problem of finding the holomorphic automorphism group of a Riemann surface given by means of an algebraic equation is not tractable in general.

Silhol presents classical and recent methods that are used in the study of the following two problems, which he calls the *uniformization problem* and the *inverse uniformization problem* respectively:

- given a Riemann surface  $S$  defined as an algebraic curve over  $\mathbb{C}$ , find its associated hyperbolic structure;
- given a discrete subgroup  $\Gamma$  of  $\mathrm{PSL}(2, \mathbb{R})$  acting on  $\mathbb{H}^2$  and satisfying certain conditions, find an algebraic curve representing the Riemann surface  $S = \mathbb{H}^2/\Gamma$ .

The methods that are used in the study of these problems involve the Schwarzian differential equation, theta functions, Poincaré series and other automorphic forms. The chapter also contains the discussion of explicit examples. The author also reports on recent work on the uniformization problem, by himself and S. Lelièvre, based on methods that were introduced by Fricke and Klein. This work concerns the uniformization of certain families of complex algebraic curves by hyperbolic surfaces obtained by gluing hyperbolic triangles or quadrilaterals along their boundaries.

Other questions related to uniformization are addressed in Chapter 18 of this volume, by Herrlich and Schmithüsen.

## 1.5 Combinatorial classes

In Chapter 5, Gabriele Mondello gives a detailed survey of the use of ribbon graphs in Teichmüller theory, in particular in the investigation of combinatorial classes in moduli space.

In this chapter,  $S$  is a compact oriented surface of genus  $g \geq 0$  equipped with a nonempty finite subset of points  $X$  of cardinality  $n$  satisfying  $2g - 2 + n > 0$ , called the marked points. As before,  $\mathcal{T}_{g,n}$  and  $\mathcal{M}_{g,n}$  denote respectively the Teichmüller and the moduli space of the pair  $(S, X)$ . A *ribbon graph* (also called a *fatgraph*) associated to  $(S, X)$  is a finite graph  $G$  embedded in  $S - X$  such that the inclusion  $G \hookrightarrow (S - X)$  is a homotopy equivalence.

Mondello describes the two main methods that have been used so far for defining ribbon graphs in the context of Teichmüller space. One definition uses complex analysis, namely, Jenkins–Strebel quadratic differentials, and the other definition uses hyperbolic geometry, more precisely, Penner’s decoration theory.

We recall that a Jenkins–Strebel differential on a Riemann surface with marked points is a meromorphic quadratic differential with at worst double poles at the marked points, whose horizontal foliation has all of its regular leaves compact. A Jenkins–Strebel differential defines a flat metric on the surface, with isolated cone singularities. The surface, as a metric space, is obtained by gluing a finite collection of Euclidean cylinders along their boundaries. The combinatorics of this cylinder decomposition of the surface is encoded by a ribbon graph.

Ribbon graphs, as they are used in Chapter 5 of this Handbook, are equipped with weights, and are called *metric ribbon graphs*. The weights, in the case just described, come from the restriction of the singular flat metric to the cylinders.

In the hyperbolic geometry approach, one considers complete finite area hyperbolic metrics on the punctured surface  $S - X$ . Neighborhoods of punctures are cusps and, around each cusp, there is a cylinder foliated by closed horocycles, that is, closed leaves whose lifts to the universal cover of  $S$  are pieces of horocycles of  $\mathbb{H}^2$ . A *decoration* on a hyperbolic punctured surface of finite area is the choice of a horocycle around each puncture. Again, these data are encoded by a metric ribbon graph.

There is a natural combinatorial structure on the space of ribbon graphs, which encodes the combinatorics of these graphs (valencies, etc.). This structure provides, via any one of the two constructions that we mentioned above, a cellularization of the space  $\mathcal{T}_{g,n} \times \Delta^{n-1}$ , where  $\Delta^{n-1}$  is the standard simplex in  $\mathbb{R}^n$ . This cellularization is invariant under the action of the mapping class group  $\Gamma_{g,n}$ , and it gives a quotient cellularization of  $\mathcal{M}_{g,n} \times \Delta^{n-1}$  (in the orbifold category). The last cellularization is one of the main tools that have been used in the study of the cohomology of moduli space and of its intersection theory. The basic work on this cellularization has been done by Harer–Mumford–Thurston, by Penner and by Bowditch & Epstein.

There is a dual object to a ribbon graph, namely, an *arc system* on the surface  $S$ . This is a collection of disjoint essential arcs with endpoints in  $X$ , which are pairwise non-homotopic with endpoints fixed.

Arc systems on the pair  $(S, X)$  naturally form a flag simplicial complex, where for each  $k \geq 0$ , a  $k$ -simplex is an arc system with  $k + 1$  components.  $\mathfrak{A}^\circ(S, X)$  denotes the *interior* of the complex  $\mathfrak{A}(S, X)$ . This is the subset of  $\mathfrak{A}(S, X)$  consisting of arc systems on  $S - X$  that cut this surface into disks or pointed disks.  $\mathfrak{A}^\infty(S, X) = \mathfrak{A}(S, X) - \mathfrak{A}^\circ(S, X)$  is called the *boundary* of  $\mathfrak{A}(S, X)$ .

Penner and Bowditch & Epstein, using decorations on hyperbolic surfaces with cusps, and Harer–Mumford–Thurston, using flat structures arising from meromorphic quadratic differentials of Jenkins–Strebel type, proved that there is a  $\Gamma_{g,n}$ -equivariant homeomorphism from the geometric realization  $|\mathfrak{A}^\circ(S, X)|$  to the product space  $\mathcal{T}_{g,n} \times \Delta^{n-1}$ . In particular, there is a  $\Gamma_{g,n}$ -equivariant homotopy equivalence  $|\mathfrak{A}^\circ(S, X)| \simeq \mathcal{T}_{g,n}$ . Via the homeomorphism  $|\mathfrak{A}^\circ(S, X)| \rightarrow \mathcal{T}_{g,n} \times \Delta^{n-1}$ , the cellular structure of  $|\mathfrak{A}^\circ(S, X)|$  is transported to  $\mathcal{T}_{g,n} \times \Delta^{n-1}$  and the homeomorphism  $|\mathfrak{A}^\circ(S, X)| \rightarrow \mathcal{T}_{g,n} \times \Delta^{n-1}$  induces a homeomorphism  $|\mathfrak{A}^\circ(S, X)|/\Gamma_{g,n} \rightarrow \mathcal{M}_{g,n} \times \Delta^{n-1}$ .

Remarkable applications of this cellularization include the following results, which are reported on by Mondello in Chapter 5 of this volume:

- Harer used this cellularization to compute the virtual cohomological dimension of the mapping class group.

- Harer & Zagier and (independently) Penner used this cellularization to compute the orbifold Euler characteristic of moduli space.<sup>4</sup>

- Kontsevich used the homeomorphism  $|\mathfrak{A}^\circ(S, X)|/\Gamma_{g,n} \rightarrow \mathcal{M}_{g,n} \times \Delta^{n-1}$  in his proof of Witten’s conjecture. Roughly speaking, the conjecture states that a certain formal power series whose coefficients are the intersection numbers of certain tautological classes on moduli space satisfies the classical KdV hierarchy of equations, that is, the generating series is a zero of certain differential operators that generate a truncated Virasoro algebra that appears in string theory.<sup>5</sup>

- Using the homeomorphism  $|\mathfrak{A}^\circ(S, X)|/\Gamma_{g,n} \rightarrow \mathcal{M}_{g,n} \times \Delta^{n-1}$ , Kontsevich, Penner and Arbarello & Cornalba studied a sequence of combinatorially defined cycles in moduli space. These cycles, called *Witten cycles*, are obtained by taking the cells that correspond to ribbon graphs with vertices of specified valencies. For instance, maximal cells correspond to trivalent ribbon graphs. Using Poincaré duality, Witten cycles define cohomology classes in  $H^{2*}(\mathcal{M}_{g,n}; \mathbb{Q})$ . Kontsevich and Penner (in different works) defined orientations on the Witten subcomplexes, Kontsevich used matrix integral techniques to express the volumes of these cycles, and Arbarello & Cornalba exploited Kontsevich’s techniques to analyze the integrals of the tautological classes over the combinatorial cycles.

- Chapter 5 also contains a sketch of a proof, obtained by Mondello and Igusa independently, of the Witten–Kontsevich conjecture (sharpened later by Arbarello & Cornalba) stating that the Witten cycles are Poincaré duals to some tautological classes defined in an algebro-geometric way on moduli space.

- Mondello introduced *generalized Witten cycles*, obtained by allowing some zero weights on the ribbon graphs that define the Witten cycles. He proved that generalized Witten cycles and tautological classes generate the same subring of  $H^*(\mathcal{M}_{g,n}; \mathbb{Q})$ . (This result was also obtained by Igusa.) Mondello also showed that there are explicit formulae that express Witten classes as polynomials in the tautological classes and

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<sup>4</sup>The enumeration methods of ribbon graphs used in their works were first developed by theoretical physicists, using asymptotic expansions of Gaussian integrals over spaces of matrices.

<sup>5</sup>A new approach to Witten’s conjecture, which is closer in spirit to the hyperbolic geometry of surfaces, has been recently developed by Maryam Mirzakhani.

vice-versa. Mondello’s proof of the Witten–Kontsevich conjecture, claiming that these cycles are polynomials in the tautological classes, provides a recursive way to find these polynomials.

The chapter also contains a discussion about the Weil–Petersson form and how the spine construction for hyperbolic surfaces with geodesic boundary interpolates between the two cellularizations of  $\mathcal{T}_{g,n} \times \Delta^{n-1}$ .

Finally, Mondello recalls Harer’s result on the stability of the cohomology groups  $H^k(\mathcal{M}_{g,n})$  for  $g > 3k$  and fixed  $n$ . Without using the result by Igusa/Mondello stated above, he exhibits a direct proof of the fact that the Witten cycles are stable. It is not clear whether similar arguments can be used for  $A_\infty$ -classes, that is, cohomology classes of  $\mathcal{M}_{g,n}$  related to certain  $A_\infty$ -algebras (Witten classes correspond to certain 1-dimensional algebras) first defined by Kontsevich, and whether these classes are tautological.

## 1.6 Differential forms

In Chapter 6, Nariya Kawazumi considers the problem of constructing “canonical” forms representing cohomology classes on moduli space. The theory is illustrated by several interesting examples, and the chapter provides an overview of various constructions of canonical two-forms.

To explain what this theory is about, Kawazumi recalls the following classical situation. Harer’s result, saying that the second homology group of the moduli space  $\mathcal{M}_g$  of a closed orientable surface  $S$  of genus  $g \geq 3$  is of rank one, implies that there exists a de Rham cohomology class which is unique up to a constant. Kawazumi’s question in that case is to find a “canonical” two-form that represents such a class. It turns out there are several such “canonical” two-forms. One non-trivial 2-cocycle for  $\mathcal{M}_g$  is the Meyer cocycle. This cocycle is related to the signature of the total space of a family of compact Riemann surfaces.

The Morita–Mumford classes are other interesting related cohomology classes. We recall that for  $n \geq 1$ , the  $n$ -th Morita–Mumford class  $e_n$  (also called tautological class) is an element of the cohomology group  $H^{2n}(\mathcal{M}_g)$ . These classes play a prominent role in the stable cohomology of the mapping class group. In 2002, I. Madsen and M. Weiss proved a conjecture that was made by Mumford, stating that the rational stable cohomology algebra of the mapping class group is generated by the Morita–Mumford classes. Kawazumi with co-authors, in a series of papers, made a deep study of the Morita–Mumford classes and their generalizations. Wolpert showed that the Weil–Petersson Kähler form  $\omega_{WP}$  represents the first Morita–Mumford class  $e_1$ . This form is an example of a “canonical” representative of  $e_1$ .

The ideas developed in Chapter 6 of this Handbook use the period map from Teichmüller space to the Siegel upper half-space. We recall that the Siegel upper half-space of genus  $g \geq 2$ , denoted by  $\mathfrak{H}_g$ , is the set of symmetric square  $g \times g$  matrices with complex coefficients whose imaginary part is positive definite. The space  $\mathfrak{H}_g$  plays an important role in number theory, being the domain of some automorphic forms (Siegel

modular forms). The period map  $\text{Jac}: \mathcal{T}_g \rightarrow \mathfrak{H}_g$  is a canonical map from Teichmüller space into the Siegel upper half-space, and the first Morita–Mumford class  $e_1$  is the pull-back of a canonical two-form on  $\mathfrak{H}_g$  by the period map. More-generally, the odd Morita–Mumford classes are represented by pull-backs of  $\text{Sp}(2g, \mathbb{R})$ -invariant differential forms on  $\mathfrak{H}_g$ , arising from Chern classes of holomorphic vector bundles. But the even ones are not. Kawazumi describes a higher analogue of the period map which he calls the *harmonic Magnus expansion*, which produces other canonical differential forms on moduli space representing the Morita–Mumford classes  $e_n$ . Some of the forms that are obtained in this way are related to Arakelov geometry.

## 2 The group theory

The group theory that is reported on in Part B of this volume concerns primarily the mapping class group of a surface. This group is studied from the point of view of quasi-homomorphisms, of measure-equivalence, and in relation to Lefschetz fibrations. Other related groups are also studied, namely, braid groups, Artin groups, and affine groups of singular flat surfaces. The study of singular flat surfaces is a subject of investigation which is part of Teichmüller theory, with ramifications in several areas in mathematics, such as dynamical systems theory, and in physics. Of particular interest in dynamical systems theory is the so-called *Teichmüller geodesic flow*, defined on the moduli space of flat surfaces.

### 2.1 Quasi-homomorphisms

Chapter 7, by Koji Fujiwara, concerns the theory of quasi-homomorphisms on mapping class groups. We recall that a quasi-homomorphism on a group  $G$  is a map  $f: G \rightarrow \mathbb{R}$  satisfying

$$\sup_{x, y \in G} |f(xy) - f(x) - f(y)| < \infty.$$

Quasi-homomorphisms on a given group form a vector space. Examples of quasi-homomorphisms are homomorphisms and bounded maps. These two classes form vector subspaces of the vector space of quasi-homomorphisms, and their intersection is reduced to the zero element.

An example of a quasi-homomorphism on  $G = \mathbb{R}$  is the integral part function, which assigns to a real number  $x$  the smallest integer  $\leq x$ .

The study of quasi-homomorphisms in relation with mapping class groups was initiated in joint work by Endo & Kotschick.<sup>6</sup>

In Chapter 7, quasi-homomorphisms on mapping class groups are studied in parallel with quasi-homomorphisms on Gromov hyperbolic groups. Although mapping

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<sup>6</sup>We note however that the case of  $\text{PSL}(2, \mathbb{Z})$ , which is the mapping class group of the torus, had already been studied by several authors.

class groups are not word-hyperbolic, since they contain subgroups isomorphic to  $\mathbb{Z}^2$  (except in some elementary cases), it is always good to find analogies between the two categories of groups. There is a well-known situation in which mapping class groups behave like generalized hyperbolic groups. This is through the action of mapping class groups on curve complexes which, by a result of Masur and Minsky, are Gromov hyperbolic. This action is co-compact but of course not properly discontinuous. Occasionally in this chapter, parallels are also made with quasi-homomorphisms on lattices in Lie groups. In the setting studied here, the techniques of proofs of corresponding results for mapping class group, hyperbolic groups and lattices present many similarities.

Using Fujiwara's notation, we let  $\widetilde{\text{QH}}(G)$  be the quotient space of the vector space of quasi-homomorphisms  $G \rightarrow \mathbb{R}$  by the subspace generated by bounded maps and by homomorphisms. The space  $\widetilde{\text{QH}}(G)$  carries a Banach space structure. One of the primary objects of the theory is to compute the vector space  $\widetilde{\text{QH}}(G)$  for a given group  $G$ , and, first of all, to find conditions under which  $\widetilde{\text{QH}}(G)$  is nonempty. It turns out that the computation of the group  $\widetilde{\text{QH}}(G)$  uses the theory of bounded cohomology. Indeed the group  $\widetilde{\text{QH}}(G)$  is the kernel of the homomorphism  $H_b^2(G; \mathbb{R}) \rightarrow H^2(G; \mathbb{R})$ , where  $H_b^2(G; \mathbb{R})$  is the second bounded cohomology group of  $G$ .

In many known cases,  $\widetilde{\text{QH}}(G)$  is either zero- or infinite-dimensional. One of the first interesting examples of the latter occurrence is due to R. Brooks, who proved in the late 1970s that in the case where  $G$  is a free group of rank  $\geq 2$ ,  $\widetilde{\text{QH}}(G)$  is infinite-dimensional.

The vector space  $\widetilde{\text{QH}}(G)$  is an interesting object associated to a hyperbolic group despite the fact that it is not a quasi-isometry invariant. Epstein & Fujiwara proved in 1997 that if  $G$  is any non-elementary word hyperbolic group, then  $\widetilde{\text{QH}}(G)$  is infinite-dimensional. Since free groups of rank  $\geq 2$  are hyperbolic, this result generalizes Brooks' result mentioned above. In 2002, Bestvina & Fujiwara extended the result of Epstein & Fujiwara to groups acting isometrically on  $\delta$ -hyperbolic spaces (with no assumption that the action is properly discontinuous). Using the action of mapping class groups on curve complexes, Bestvina & Fujiwara proved that if  $G$  is any subgroup of the mapping class group of a compact orientable surface which is not virtually abelian, then  $\widetilde{\text{QH}}(G)$  is infinite-dimensional.

Chapter 7 contains a review of these results as well as a short introduction to the theory of bounded cohomology for discrete groups. The author also surveys some recent results by Bestvina & Fujiwara on the group  $\widetilde{\text{QH}}(G)$  in the case where  $G$  is the fundamental group of a complete Riemannian manifold of non-positive sectional curvature. He describes some rank-one properties of mapping class groups related to quasi-homomorphisms, to some superrigidity phenomena and to the bounded generation property. We recall that a group  $G$  is said to be *boundedly generated* if there exists a finite subset  $\{g_1, \dots, g_k\}$  of  $G$  such that every element of this group can be written as  $g_1^{n_1} \dots g_k^{n_k}$  with  $n_1, \dots, n_k$  in  $\mathbb{Z}$ . Bounded generation is related to the existence of quasi-homomorphisms. Mapping class groups are not boundedly generated (Farb–

Lubotzky–Minsky). Non-elementary subgroups of word-hyperbolic groups are not boundedly generated (Fujiwara). A discrete subgroup of a rank-1 simple Lie group that does not contain a nilpotent subgroup of finite index is not boundedly generated (Fujiwara).

Chapter 7 also contains a survey of the theory of separation by quasi-homomorphisms in groups, with applications to mapping class groups, to hyperbolic groups and to lattices. One of the motivating results in this direction is a result by Polterovich & Rudnick (2001) saying that if two elements in  $\mathrm{SL}(2, \mathbb{Z})$  are not conjugate to their inverses, then they can be separated by quasi-homomorphisms. Recent results on this subject, by Endo & Kotschick for mapping class groups and by Calegari & Fujiwara for hyperbolic groups, are presented in this chapter.

## 2.2 Lefschetz fibrations

Chapter 8, by Mustafa Korkmaz and András Stiepicz, concerns the theory of Lefschetz pencils and Lefschetz fibrations, a theory which is at the intersection of 4-manifold theory, algebraic geometry and symplectic topology. Mapping class groups of surfaces play an essential role in this theory, and it is for this reason that such a chapter is included in this Handbook.

Lefschetz fibrations are 4-dimensional manifolds that are simple enough to handle, but with a rich enough structure to make them interesting. One may consider a Lefschetz fibration as a natural generalization of a 4-manifold which is a surface fibration, a surface fibration being itself a generalization of a Cartesian product of two surfaces. Lefschetz pencils are slightly more general than Lefschetz fibrations; a Lefschetz pencil gives rise to a Lefschetz fibration by a “blowing-up” operation.

Lefschetz fibrations and Lefschetz pencils first appeared in algebraic geometry in the early years of the twentieth century, when Solomon Lefschetz studied such structures on complex algebraic surfaces, that is, 4-dimensional manifolds defined as zeroes of a homogeneous polynomial systems with complex coefficients. Lefschetz constructed a Lefschetz pencil structure on every algebraic surface.

Towards the end of the 1990s, Lefschetz fibrations and Lefschetz pencils played an important role in the work of Simon Donaldson, who showed that any symplectic 4-manifold has a Lefschetz pencil structure with base the two-sphere. Robert Gompf showed that conversely, any 4-manifold admitting a Lefschetz pencil structure carries a symplectic structure.<sup>7</sup> In this way, Lefschetz pencils play the role of a topological analogue of symplectic 4-manifolds.

Let us say things more precisely. A *Lefschetz fibration* is a compact oriented 4-dimensional manifold  $X$  equipped with a projection  $\pi : X \rightarrow S$ , where  $S$  is a closed oriented surface, and where  $\pi$  is a fibration if we restrict it to the inverse image of some finite set of points in  $S$ , called the critical values. Furthermore, it is required that

<sup>7</sup>Gompf’s proof is an extension to the class of Lefschetz pencils of Thurston’s proof of the fact that any oriented surface bundle over a surface carries a symplectic structure, provided that the homology class of the fiber is nontrivial in the second homology group of the 4-manifold.

above a critical value, the local topological model of  $\pi$  is the map  $(z_1, z_2) \mapsto z_1^2 + z_2^2$  from  $\mathbb{C}^2$  to  $\mathbb{C}$ , in the neighborhood of the origin. (In this picture, the critical value is the origin.) The fibers of  $\pi$  above the critical values are singular surfaces, and a singular point on such a surface is called a *nodal point*. A nonsingular fiber is a closed orientable surface called a *generic fiber*. The *genus* of a Lefschetz fibration is, by definition, the genus of a regular fiber. (Recall that restricted to the complement of the critical values, a Lefschetz fibration is a genuine fibration, and therefore all the generic fibers are homeomorphic.) In some sense, a nodal point is a singularity of the simplest type in the dimension considered; it is the singularity that appears at a generic intersection of two surfaces. Such a singularity naturally appears in complex analysis. In a Lefschetz fibration, a singular fiber is obtained from a nearby fiber by collapsing to a point a simple closed curve, called a *vanishing cycle*. The vanishing cycle, when it is collapsed, becomes the nodal point of the corresponding singular fiber.

A natural way of studying the topology of a Lefschetz fibration  $\pi : X \rightarrow S$  is to try to figure out how the fibers  $\pi^{-1}(s)$  are glued together in  $X$  when the point  $s$  moves on the surface  $S$ , and in particular, near the critical values, since the complication comes from there. This leads to a combinatorial problem which in general is non-trivial, and the mapping class group of a generic fiber is an essential ingredient in this story. It is here that the study of Lefschetz fibrations gives rise to interesting problems on mapping class groups. For instance, Lefschetz fibrations were the motivation of recent work by Endo & Kotschick and by Korkmaz on commutator lengths of elements in mapping class groups. Lefschetz fibrations also motivated the study of questions related to “factorizations of the identity element” of a mapping class group, that is, an expansion of this identity as a product of positive Dehn twists.

I would like to say a few words on monodromies and on factorizations, and this needs some notation.

Let  $P \subset S$  be the set of critical values of a Lefschetz fibration  $\pi : X \rightarrow S$ . We choose a basepoint  $s_0$  for the surface  $S$ , in the complement of the set  $P$ . The fiber  $\pi^{-1}(s_0)$  is then called the *base fiber* and we identify it with an abstract surface  $F$ . There is a natural homomorphism  $\psi$ , called the *monodromy representation* from  $(\pi_1(S - P), s_0)$  to the mapping class group of  $F$ . This homomorphism is the main algebraic object that captures the combinatorics of the Lefschetz fibration. It is defined by considering, for each loop  $\gamma : [0, 1] \rightarrow S$  based at  $s_0$ , the fibration induced on the interval  $[0, 1]$  (which is a trivial fibration), and then taking the isotopy class of the surface homeomorphism that corresponds to the gluing between the fibers of  $\pi$  above the points  $\gamma(0)$  and  $\gamma(1)$ . The resulting *monodromy representation* is a homomorphism  $\psi$  from  $(\pi_1(S - P), s_0)$  to the mapping class group of  $F$ , and it is well defined up to conjugacy. Two Lefschetz fibrations are isomorphic if and only if they have the same monodromy representation (up to an isomorphism between the images induced by inner automorphisms of the mapping class groups of the fibers, and up to isomorphisms of the fundamental groups of the bases of the fibrations). The detailed construction of the monodromy representation is recalled in Chapter 8 of this volume.



The monodromy representation homomorphism in this theory can be compared to the monodromy which appears in the study of holomorphic families of Riemann surfaces, as it is presented in Chapter 3 of this volume.

Now a few words about factorizations. The monodromy around a critical value is the class of the positive Dehn twist along the vanishing cycle on a regular fiber near the singular fiber. Modulo some standard choices and identifications, the monodromy associated to a loop that surrounds exactly one time each critical value produces an element of the mapping class group of the base fiber, which is equal to the identity word decomposed as a product of positive Dehn twists. Conversely, one can construct a Lefschetz fibration of genus  $g$  from each factorization of the identity element of the mapping class group of an oriented closed surface of genus  $g$ . There is an action of the braid group on the set of such factorizations, and the induced equivalence relation is called *Hurwitz equivalence*. The notion of factorization in this setting leads to a discussion of commutator length and of torsion length in the mapping class group. More precisely, it leads to the question of the minimal number of factors needed to express an element of the mapping class group as a product of commutators and of torsion elements respectively.

This chapter by Korkmaz and Stiepicz gives a quick overview on Lefschetz fibrations, with their relation to the works of Gompf and Donaldson on symplectic topology, and to the works of Endo & Kotschick and of Korkmaz on commutator lengths of Dehn twists in mapping class groups. The authors also mention generalizations of Lefschetz fibrations involving Stein manifolds and contact structures. They propose a list of open problems on the subject.

## 2.3 Measure-equivalence

Chapter 9, by Yoshikata Kida, considers mapping class groups in analogy with lattices, that is, discrete subgroups of cofinite volume of Lie groups, in the special setting of group actions on measure spaces.

Lattice examples are appealing for people studying mapping class groups, because it is a natural question to search for properties of mapping class groups that are shared by lattices, and for properties of mapping class groups that distinguish them from lattices. We already mentioned these facts in connection with Fujiwara's work in Chapter 7, and we recall in this respect that  $\mathrm{PSL}(2, \mathbb{Z})$ , which is the mapping class group of the torus, is a lattice in  $\mathrm{PSL}(2, \mathbb{R})$ .

At the same time, Chapter 9 gives a review of measure-equivalence theory applied to the study of mapping class groups.

Let us first recall a few definitions. Two discrete groups  $\Gamma$  and  $\Lambda$  are said to be *measure-equivalent* if there exists a standard Borel space  $(\Sigma, m)$  (that is, a Borel space equipped with a  $\sigma$ -finite positive measure which is isomorphic to a Borel subset of the unit interval) equipped with a measure-preserving action of the direct product  $\Gamma \times \Lambda$ , such that the actions of  $\Gamma$  and  $\Lambda$  obtained by restricting the  $\Gamma \times \Lambda$ -action to  $\Gamma \times \{e\}$  and  $\{e\} \times \Lambda$  satisfy the following two properties:

- these actions are essentially free, that is, stabilizers of almost all points are trivial;
- these actions have finite-measure fundamental domains.

Measure-equivalence is an equivalence relation on the class of discrete groups. It was introduced by Gromov in his paper *Asymptotic invariants*, as a measure-theoretic analogue of quasi-isometry, the latter being defined on the class of finitely generated groups. Gromov raised the question of classifying discrete groups up to measure-equivalence.

From the definitions, it follows easily that isomorphic groups modulo finite kernels and co-kernels are measure-equivalent. In particular, any two finite groups are measure-equivalent. A group that is measure-equivalent to a finite group is finite. In any locally compact second countable Lie group, two lattices are measure-equivalent.

Two discrete groups  $\Gamma$  and  $\Lambda$  acting on two standard measure spaces  $(X, \mu)$  and  $(Y, \nu)$  are said to be *orbit-equivalent* if there exists a measure-preserving isomorphism  $f: (X, \mu) \rightarrow (Y, \nu)$  such that  $f(\Gamma x) = \Lambda f(x)$  for almost every  $x$  in  $X$ . Orbit-equivalence is an equivalence relation which is weaker than conjugacy, and it is intimately related to measure-equivalence. The study of orbit-equivalence was started a few decades ago by D. S. Ornstein and B. Weiss. These authors showed that an infinite discrete group is measure-equivalent to  $\mathbb{Z}$  if and only if it is amenable. Their result was stated in terms of orbit-equivalence. Orbit-equivalence is also related to the study of von Neumann algebras, and it was studied as such by S. Popa.

In a series of recent papers, Y. Kida made a detailed study of measure-equivalence in relation to mapping class groups. In particular, he obtained the following results, reported on in Chapter 9 of this volume.

Let  $S = S_{g,p}$  be a compact surface of genus  $g$  with  $p$  boundary components satisfying  $3g - 4 + p > 0$  and let  $C(S)$  be the curve complex of  $S$ . If a discrete group  $\Lambda$  is measure-equivalent to the mapping class group of  $S$ , then there exists a homomorphism  $\rho: \Lambda \rightarrow \text{Aut}(C(S))$  whose kernel and cokernel are both finite. Using the famous result by Ivanov (completed by Korkmaz and Luo) stating that (with a small number of exceptional surfaces) the automorphism group of the curve complex of a surface is the extended mapping class group of that surface, Kida's result gives a characterization of discrete groups that are measure-equivalent to mapping class groups. This result is an analogue of a result by A. Furman which gives a characterization of discrete groups that are measure-equivalent to higher rank lattices.

Kida also studied the relation of measure-equivalence between surface mapping class groups, proving that if two pairs of nonnegative integers  $(p, g)$  and  $(p', g')$  satisfy  $3g - 4 + p \geq 0$  and  $3g' - 4 + p' \geq 0$ , and if the mapping class groups  $\Gamma(S_{g,p})$  and  $\Gamma(S_{g',p'})$  are measure-equivalent, then either the surfaces  $S_{g,p}$  and  $S_{g',p'}$  are homeomorphic or  $\{(g, p), (g', p')\}$  is equal to  $\{(0, 5), (1, 2)\}$  or to  $\{(0, 6), (2, 0)\}$ . He also settled the question of the classification of subgroups of mapping class groups from the viewpoint of measure-equivalence. An analogous result was known for lattices in the Lie groups  $\text{SL}(n, R)$  and  $\text{SO}(n, 1)$ .

Kida showed that there exist no interesting embedding of the mapping class group as a lattice in a locally compact second countable group. V. Kaimanovich and H. Masur

had already proved that under the condition  $3g - 4 + p \geq 0$ , any sufficiently large subgroup of the mapping class group of  $S_{g,p}$  (and in particular, the mapping class group itself) is not isomorphic to a lattice in a semisimple Lie group with real rank at least two.

Inspired by a definition made by R. Zimmer in the setting of lattices, Kida defined a notion of measure-amenability for actions on the curve complex of a surface. He proved the following: Let  $S = S_{g,p}$  be a surface satisfying  $3g - 4 + p \geq 0$ , let  $C(S)$  be the curve complex of  $S$ , let  $\partial C(S)$  be its Gromov boundary and let  $\mu$  be a probability measure on  $\partial C(S)$  such that the action of the extended mapping class group of  $S$  on that measure space is non-singular. Then this action is measure-amenable.

Chapter 9 also contains interesting measure-theoretic descriptions of mapping class group actions, e.g., a classification of infinite subgroups of the mapping class group in terms of the fixed points of their actions on the space of probability measures on Thurston's space of projective measured foliations.

It is interesting to see that Y. Kida succeeded in replacing by measure-theoretic arguments the topological arguments that were used by various authors in the proofs of their rigidity results on mapping class group actions on several spaces (e.g., the actions on the curve complex and on other complexes, the actions on spaces of foliations, algebraic actions of the extended mapping class group on itself by conjugation, and so on). To give an example that highlights the analogy, we recall a result by N. Ivanov stating that, with the exception of some special surfaces, any isomorphism  $\phi: \Gamma_1 \rightarrow \Gamma_2$  between finite index subgroups  $\Gamma_1$  and  $\Gamma_2$  of the extended mapping class group is a conjugation by an element of the extended mapping class group, and in particular, any automorphism of the extended mapping class group is an inner automorphism. An important step in Ivanov's proof of this result is the proof that any automorphism between  $\Gamma_1$  and  $\Gamma_2$  sends a sufficiently high power of a Dehn twist to a power of a Dehn twist. From this, and since Dehn twists are associated to homotopy classes of simple closed curves which are vertices of the curve complex, Ivanov obtains an automorphism of the curve complex induced by the isomorphism  $\phi$ . He then appeals to the fact that the automorphism group of the curve complex is the natural image in that group of the extended mapping class group. To prove that  $\phi$  sends powers of Dehn twists to powers of Dehn twists, Ivanov uses an algebraic characterization of Dehn twists. Moreover, he proves that  $\phi$  preserves some geometric relations between Dehn twists; for instance, it sends pairs of commuting Dehn twists to pairs of commuting Dehn twists. Now the measure-theoretic setting. Kida's rigidity result is formulated in the general setting of isomorphisms of discrete measured groupoids. To say it in few words, Kida needs to show that any isomorphism of discrete measured groupoids arising from measure-preserving actions of the mapping class group preserves subgroupoids generated by Dehn twists. The proof of this fact uses a characterization of such groupoids in terms of discrete measured groupoid invariants. This is done by using the measure-amenability of non-singular actions of the extended mapping class group on the boundary of the curve complex mentioned above, and a subtle characterization of subgroupoids generated by Dehn twists in terms of measure-amenability.

More precisely, a subgroupoid generated by a Dehn twist is characterized by the fact that it is an amenable normal subgroupoid of infinite type of some maximal reducible subgroupoid. Kida concludes using the fact that measure-amenability is an invariant of isomorphism between groupoid actions.

Kida also obtained a measurable rigidity result for direct products of mapping class groups, using a technique introduced by N. Monod and Y. Shalom in a study they made of measurable rigidity of direct products of discrete groups.

Recently, D. Gaboriau showed that the sequence of  $\ell^2$ -Betti numbers introduced by Cheeger and Gromov is invariant under measure-equivalence, up to a multiplicative constant. Using this and results of McMullen and of Gromov, Kida gave formulae for these Betti numbers.

## 2.4 Affine groups

In Chapter 10, a *flat surface* is defined as a pair  $(S, \omega)$  consisting of a closed Riemann surface  $S$  equipped with a nonzero holomorphic one-form  $\omega$  (which we shall also call here an abelian differential). Such a surface  $S$  is naturally equipped with a flat (i.e. Euclidean) structure in the complement of the zeroes of  $\omega$ . The flat structure is defined, using the holomorphic local coordinates, by parameters of the form  $\phi(z) = \int_{z_0}^z \omega$ , after a choice of a basepoint  $z_0$  in the holomorphic chart. In fact, the surface  $S$  is equipped, in the complement of the zeroes of  $\omega$ , with an atlas whose transition functions are better than Euclidean transformations of the plane, since they are translations. For this reason, a flat surfaces in the sense used here is also called a “translation surface”. The flat metric in the complement of the zeroes of  $\omega$  extends at any zero point of order  $n$  to a singular flat metric whose singularity at such a point is locally a Euclidean cone point with total angle  $2\pi(n + 1)$ . We note that there are other ways of defining flat surfaces that do not use the word “holomorphic”. For instance, a flat surface can be obtained by gluing rational-angled Euclidean polygons along their boundaries by Euclidean translations.

There is a strong relation between flat surfaces and billiards. In 1975, Zelmyakov & Katok associated to each rational-angled polygon a uniquely defined flat surface, such that the billiard flow of the polygon is equivalent to the geodesic flow of the flat surface.

There is a natural action of the group  $\mathrm{SL}(2, \mathbb{R})$  on the space of flat surfaces, and this action preserves the space  $\mathcal{A}$  of unit norm abelian differentials (the norm of a flat surface  $(S, \omega)$  being defined by  $(\int_S |\omega|^2)^{1/2}$ ). We also recall that the *Teichmüller geodesic flow* is the action of the diagonal subgroup of  $\mathrm{SL}(2, \mathbb{R})$  on the space  $\mathcal{A}$ .

Flat surfaces appear in many ways in Teichmüller theory. One obvious reason is that a flat surface has an underlying Riemann surface structure, and it is therefore natural to study parametrizations of Teichmüller space by flat surfaces. Flat surfaces also arise from holomorphic quadratic differentials. We recall that a holomorphic quadratic differential being locally the square of a holomorphic one-form, also gives rise to a singular Euclidean metric on its underlying Riemann surface. Holomorphic

quadratic differentials play a prominent role in Teichmüller theory since the work of Teichmüller himself, in particular because there is a natural identification between the vector space of quadratic differentials and the cotangent space to Teichmüller space at each point.

To a flat surface  $(S, \omega)$  is associated a subgroup of  $SL(2, \mathbb{R})$  called its *affine group*, and denoted by  $SL(S, \omega)$ . To define this group, one first considers the group  $\text{Aff}^+(S, \omega)$  of orientation-preserving diffeomorphisms of  $S$  that act affinely in the Euclidean charts associated to  $\omega$ , in the complement of the zeroes of  $\omega$ . (Such a diffeomorphism is allowed to permute the zeroes.) An affine map, in a chart, has a matrix form  $X \mapsto AX + B$ , with  $A$  being a constant nonsingular matrix which can be considered as the derivative of the affine map. Since the coordinate changes of the Euclidean atlas associated to a flat surface are translations, the matrix  $A$  is independent of the choice of the chart, and thus is canonically associated to the affine map. Composing two affine diffeomorphisms of  $S$  gives rise to matrix multiplication at the level of the linear parts. This gives a homomorphism  $D: \text{Aff}^+(S, \omega) \rightarrow GL(2, \mathbb{R})$  which associates to each affine diffeomorphism its derivative. The image of  $D$  lies in the subgroup  $SL(2, \mathbb{R})$  of  $GL(2, \mathbb{R})$ , as a consequence of the fact that the surface has finite area. The image of the diffeomorphism  $D$  in  $SL(2, \mathbb{R})$  is, by definition, the *affine group*  $SL(S, \omega)$  of the flat surface  $(S, \omega)$ . W. Veech observed that the affine group  $SL(S, \omega)$  is always a discrete subgroup of  $SL(2, \mathbb{R})$ . The affine group  $SL(S, \omega)$  is sometimes called the *Veech group* of the flat surface.

There is a nice description of Thurston's classification of isotopy classes of affine diffeomorphism. An affine homeomorphism  $f: S \rightarrow S$  is parabolic, elliptic or hyperbolic if  $|\text{Tr}(Df)| = 2, < 2, \text{ or } > 2$  respectively. The hyperbolic affine homeomorphisms are the pseudo-Anosov affine diffeomorphisms. Beyond their use in this classification, we shall see below that the set of traces of affine homeomorphisms of a flat surface play a special role in this theory.

The notion of an affine group of a flat surface first appeared in Thurston's construction of a family of pseudo-Anosov homeomorphisms of a surface which are affine with respect to some flat structure. Indeed, in his paper *On the geometry and dynamics of homeomorphisms of surfaces*, Thurston constructed such a family, the flat structure being obtained by "thickening" a filling pair of transverse systems of simple closed curves on the surface.

In Chapter 10 of this Handbook, Martin Möller addresses the following natural problems:

- Which subgroups of  $SL(2, \mathbb{R})$  arise as affine groups of flat surfaces?
- What does the affine group of a generic flat surface look like?

Several partial results on these problems have been obtained by various authors. For instance, Veech constructed flat surfaces whose affine groups are non-arithmetic lattices. Special types of flat surfaces, called origamis, or square-tiled surfaces, arise naturally in these kinds of questions. These surfaces are obtained by gluing Euclidean squares along their boundaries using Euclidean translations. E. Gutkin & C. Judge showed that the affine group of an origami is a subgroup of finite index in  $SL(2, \mathbb{Z})$ .

P. Hubert & S. Lelièvre showed that in any genus  $g \geq 2$  there are origamis whose affine groups are non-congruence subgroups of  $SL(2, \mathbb{R})$ . We note that origamis were already considered in Volume I of this Handbook, namely in Chapter 6 by Herrlich and Schmithüsen, where these surfaces are studied in connection with Teichmüller disks in moduli space. They are also thoroughly studied in relation with the theory of dessins d'enfants in Chapter 18 of the present volume. Schmithüsen proved that all congruence subgroups of  $SL(2, \mathbb{Z})$  with possibly five exceptions occur as affine groups of origamis. Möller, in Chapter 10 of this volume, asks the question of whether there is a subgroup of  $SL(2, \mathbb{Z})$  that is not the affine group of an origami.

Another interesting class of flat surfaces is the class of *Veech surfaces*. These are the flat surfaces whose affine groups are lattices in  $SL(2, \mathbb{R})$ . A recent result of I. Bouw and M. Müller says that all triangle group  $(m, n, \infty)$  with  $1/m + 1/n < 1$  and  $m, n \leq \infty$  occur as affine groups of Veech surfaces.

C. McMullen, and then P. Hubert & T. Schmidt produced flat surfaces whose affine groups are infinitely generated.

Möller proved that provided the genus of  $S$  is  $\geq 2$ , the affine group of a generic flat surface  $(S, \omega)$  is either  $\mathbb{Z}/2$  or trivial, and that this depends on whether  $(S, \omega)$  is in a hyperelliptic component or not, with respect to the natural stratification of the total space of the vector bundle of holomorphic one-forms minus the zero-section. (A hyperelliptic component is a component of a stratum that consists exclusively of hyperelliptic curves.) He also proved that in every stratum there exist flat surfaces whose affine groups are cyclic groups generated by parabolic elements. He raises the question of whether there exists a flat surface whose affine group is cyclic generated by a hyperbolic element.

Müller also discusses the relation between affine groups and closures of  $SL(2, \mathbb{R})$ -orbits of the corresponding flat surfaces in moduli space.

Given an arbitrary subgroup  $\Gamma$  of  $SL(2, \mathbb{R})$ , one can define its *trace field* as the subfield  $K$  of  $\mathbb{R}$  generated by the set  $\{\text{Tr}(A) : A \in \Gamma\}$ . Thus, associated to a flat surface  $(S, \omega)$  is the trace field of its affine group  $SL(S, \omega)$ . It turns out that the trace field of the affine group of a flat surface is an interesting object of study. R. Kenyon & J. Smillie proved that the trace field of the affine group  $SL(S, \omega)$  has at most degree  $g$  over  $\mathbb{Q}$ . P. Hubert & E. Lanneau showed that if  $(S, \omega)$  is given by Thurston's construction, then the trace field of  $SL(S, \omega)$  is totally real. They also showed that there exist flat surfaces supporting pseudo-Anosov diffeomorphisms whose trace fields are not totally real. C. McMullen showed that all real quadratic fields arise as trace fields of lattice affine groups.

## 2.5 Braid groups

Chapter 11 by Luis Paris is a survey on braid groups and on some of their generalizations, and on the relations between these groups and mapping class groups.

Braid groups are related to mapping class groups in several ways. A well-known instance of such a relation is that the braid group on  $n$  strands is isomorphic to the

mapping class group of the surface  $S_{0,n}$ , that is, the disk with  $n$  punctures. In fact, this isomorphism can be considered as a first step for a general theory of representations of braid groups in mapping class groups, which is one of the main subjects reported on in Chapter 11.

Although braiding techniques have certainly been known since the dawn of humanity (hair braiding, rope braiding, etc.), braid groups as mathematical objects were formally introduced in 1925, by Emil Artin, and questions about representations of braid groups immediately showed up. One of the first important results in this representation theory is due to Artin himself, who proved that the braid group on  $n$  strands admits a faithful representation (now called the *Artin representation*) in the automorphism group of the free group on  $n$  generators.<sup>8</sup> Artin's result can be seen as an analogue of the result by Dehn, Nielsen and Baer stating that the extended mapping class group of a closed surface of genus  $\geq 1$  admits a faithful representation in the automorphism group of the fundamental group of that surface (and in that case, the representation is an isomorphism). From Artin's result one deduces immediately that braid groups are residually finite and Hopfian. (Recall that a group is said to be Hopfian if it is not isomorphic to any of its subgroups.)

Historically, results on braid groups were obtained in general before the corresponding results on mapping class groups. This is due to the fact that braid groups have very simple presentations, with nothing comparable in the case of mapping class groups. Another possible reason is that homeomorphisms of the punctured disk are much easier to visualize compared to homeomorphisms of arbitrary surfaces, and therefore, it is in principle easier to have a geometric intuition on braid groups than on general surface mapping class groups. It is also safe to say that results on braid groups have inspired research on mapping class groups. Indeed, several results on mapping class groups were conjectured in analogy with results that were already obtained for braid groups. Let us mention a few examples:

- Presentations of braid groups have been known since the introduction of these groups. (In fact, right at the beginning, braid groups were defined by generators and relators.) But in the case of the mapping class groups, it took several decades after the question was addressed, to find explicit presentations.
- Automorphism groups of braid groups were computed long before analogous results were obtained for mapping class groups.
- Several algorithmic problems (conjugacy and word problems, etc.) were solved for braid groups before results of the same type were obtained for mapping class groups.
- The existence of a faithful linear representation for braid groups has been obtained in the year 2000 (by Bigelow and Krammer, independently), settling a question that had been open for many years. The corresponding question for mapping class groups is still one of the main open questions in the field.

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<sup>8</sup>B. Perron and J. P. Vannier recently obtained results on the representation of a braid group on  $n$  strands in the automorphism group of the free group on  $n - 1$  generators.

In Chapter 11 of this volume, the theory of braids is included in a very wide setting that encompasses mapping class groups, but also other combinatorially defined finitely presented groups, namely Garside groups, Artin groups and Coxeter groups. To make things more precise, we take a finite set  $S$  of cardinality  $n$  and we recall that a *Coxeter matrix* over  $S$  is an  $n \times n$  matrix whose coefficients  $m_{st}$  ( $s, t \in S$ ) belong to the set  $\{1, 2, \dots, \infty\}$ , with  $m_{st} = 1$  if and only if  $s = t$ . The *Coxeter graph*  $\Gamma$  associated to a Coxeter matrix  $M = m_{s,t}$  is a labeled graph whose vertex set is  $S$  and where two distinct vertices  $s$  and  $t$  are joined by an edge whenever  $m_{s,t} \geq 3$ . If  $m_{st} \geq 4$ , then the edge is labeled by  $m_{s,t}$ . Coxeter graphs are also called Dynkin diagrams. The *Coxeter group of type*  $\Gamma$  is the finitely presented group with generating set  $S$  and relations  $s^2 = 1$  for  $s$  in  $S$ , and  $(st)^{m_{st}} = 1$  for  $s \neq t$  in  $S$ . Here, a relation with  $m_{st} = \infty$  means that the relation does not exist.

The *Artin group* associated to a Coxeter matrix  $M = m_{s,t}$  is a group defined by generators and relations, where the generators are the elements of  $S$ , ordered as a sequence  $\{a_1, \dots, a_n\}$  and where the relations are defined by the equalities  $\langle a_1, a_2 \rangle^{m_{1,2}} = \langle a_2, a_1 \rangle^{m_{2,1}}, \dots, \langle a_{n-1}, a_n \rangle^{m_{n-1,n}} = \langle a_n, a_{n-1} \rangle^{m_{n,n-1}}$  for all  $m_{i,j} \in \{2, 3, \dots, \infty\}$ , where  $\langle a_i, a_j \rangle$  denotes the alternating product of  $a_i$  and  $a_j$  taken  $m_{i,j}$  times, starting with  $a_i$ . (For example,  $\langle a_1, a_2 \rangle^5 = a_1 a_2 a_1 a_2 a_1$ .) Artin groups are also used in other domains of mathematics, for instance in the theory of random walks.

Coxeter groups were introduced by J. Tits in relation with his study of Artin groups. Garside groups were introduced by P. Dehornoy and L. Paris, as a generalization of Artin groups. There are several relations between Artin groups, Coxeter groups and Garside groups. One important aspect of Garside groups is that these groups are well-suited to the study of algorithmic problems for braid groups. An Artin group has a quotient Coxeter group.

There is a geometric interpretation of Artin groups which extends the interpretation of braid groups in terms of fundamental groups of hyperplane arrangements in  $\mathbb{C}^n$ . It is unknown whether mapping class groups are Artin groups and whether they are Garside groups. Some Artin groups, called Artin groups of spherical type, are Garside groups, and it is known that Artin groups of spherical type are generalizations of braid groups.

Chapter 11 contains algebraic results, algorithmic results, and results on the representation theory of these classes of groups.

From an algebraic point of view, Paris gives an account of known results on the cohomology of braid groups and of Artin groups of spherical type. He introduces Salvetti complexes of hyperplane arrangements. These complexes are simplicial complexes that arise naturally in the study of hyperplane arrangements; they have natural geometric realizations, and they have been successfully used as a tool in computing the cohomology of Artin groups.

From the algorithmic point of view, the author reports on Tits' solution of the word problem for Coxeter groups, on Garside's solution of the conjugacy problem for braid groups, and on recent progress made by Dehornoy and Paris on the extension of this result to Garside groups.



Paris also reports on recent progress on linear representations of Artin groups, extending the work by Bigelow and Krammer on linear representations of braid groups and the subsequent work on linear representations of certain Artin groups, which was done by Digne and by Cohen & Wales. The author also presents an algebraic and a topological approach that he recently developed for the question of linear representations.

Besides the study of linear representations, Chapter 11 contains a recent study of *geometric* representations of Artin groups, that is, representations into mapping class groups. (Recall the better-than-faithful representation of the braid group on  $n$  strands in the mapping class group of the disk with  $n$  punctures.) The chapter contains the description of a nice construction of geometric representations of Artin groups, obtained by sending generators to Dehn twists along some curves that realize the combinatorics of the associated Coxeter graph.

### 3 Representation spaces and geometric structures

Representation theory makes interesting relations between algebra and geometry. From our point of view, the subject may be described as the study of geometric structures by representing them by matrices and algebraic operations on these matrices.

As already mentioned, the geometric structures considered in Part C of this Handbook are more general than the structures that are dealt with in the classical Teichmüller theory (namely, conformal structures and hyperbolic structures). These general structures include complex projective structures, whose recent study involves techniques that have been introduced by Thurston in the 1990s. We recall that Thurston introduced parameters for (equivalence classes of) complex projective structures on a surface in which the space of measured laminations plays an essential role. In this setting, complex projective structures are obtained by *grafting* Euclidean annuli on hyperbolic surfaces along simple closed curves and, more generally, along measured geodesic laminations. As it is the case for hyperbolic structures, deformations of complex projective structures can be studied either directly on the surface, or within a space of representations of the fundamental group of the surface in an appropriate Lie group. The direct study can be done by considering complex projective structures defined on some “elementary” surfaces with boundary and then gluing together such surfaces so as to obtain complex projective structures on larger surfaces. For instance, one can study complex projective structures on pairs of pants in a way parallel to what is done classically in the study of hyperbolic structures, and then investigate the gluing between pairs of pants. Complex projective structures can also be studied in the context of representations of fundamental groups of surfaces in the Lie group  $SL(2, \mathbb{C})$ . It is also well known that the space of  $SL(2, \mathbb{C})$ -representations (more precisely, the orbit space under the action of  $SL(2, \mathbb{C})$  by conjugation) can also be studied for itself, as a generalized Teichmüller space.

In this part of the Handbook, Bill Goldman gives an exposition of what is usually referred to as the *Fricke–Klein trace parameters*, that is, parameters for the representations of Teichmüller spaces in the character variety of  $\mathrm{SL}(2, \mathbb{C})$ -representations. He treats the cases of surfaces with two and three-generator fundamental groups in full detail. The parameters that are given are explicit. Although the subject is very classical, such a complete study is done for the first time. Also in this part, Sean Lawton and Elisha Peterson develop a diagrammatic approach to the study of the structure of the  $\mathrm{SL}(2, \mathbb{C})$ -representations character variety for the free group on two generators using graphs that are called spin networks. Their work sits in the framework of geometric invariant theory, a theory that develops the idea (first started by Vogt and Fricke–Klein) of characterizing polynomial functions on  $\mathrm{SL}(2, \mathbb{C})$  that are invariant under inner automorphisms, and that are expressible in terms of traces.

Another generalized Teichmüller space that is considered in this volume is the space of Lorentzian 3-manifolds of constant curvature which are products of surfaces with the real line, in which Thurston’s hyperbolic geometry techniques (laminations, earthquakes, grafting and so on) were brought in in the 1990s by Geoffrey Mess.

### 3.1 Complex projective structures

As already said, the study of moduli of complex projective structure is intimately related to that of Teichmüller space. Thus it is natural to include in the Handbook a chapter on complex projective structures.

Complex projective structures on surfaces already appear in a substantial manner in the work of Poincaré. The relation between complex projective structures on surfaces and Teichmüller theory was developed by Bers and his collaborators in the 1960s. For instance, the Bers embedding of Teichmüller space can be described in terms of complex projective structures.

The model space for complex projective geometry on surfaces is the complex projective line  $\mathbb{CP}^1$ , that is, the space of 1-dimensional complex vector subspaces of  $\mathbb{C}^2$ , with transformation group induced from the linear transformations of  $\mathbb{C}^2$ . Equivalently, we can consider the model space of complex projective geometry on surfaces as the Riemann sphere  $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$  equipped with the group of transformations of the form  $z \mapsto \frac{az+b}{cz+d}$  with  $a, b, c, d \in \mathbb{C}$  and  $ad - bc \neq 0$ . Such transformations are called *fractional linear transformations*, or *Möbius transformations*, or *projective transformations*. A complex projective structure on a surface is then an atlas with charts in  $\mathbb{CP}^1$  whose coordinate changes are restrictions of projective transformations. Markings of complex projective surfaces are defined as it is usually done in Teichmüller theory, that is, a marking is a homeomorphism from a fixed base surface to a surface equipped with a projective structure. There is a natural equivalence relation on the set of marked projective structures, defined, again as in Teichmüller theory, by the existence of a projective transformation in the correct homotopy class. If  $S$  is a closed surface of genus  $\geq 2$ , we shall denote by  $\mathcal{P}(S)$  the space of equivalence classes of marked complex projective structures on  $S$ .

In what follows, “projective” means “complex projective”.

Since the projective transformations of the sphere are holomorphic, a projective structure on a surface has an underlying conformal structure. In other words, there is a *forgetful map*  $\pi : \mathcal{P}(S) \rightarrow \mathcal{T}(S)$  from the space of marked projective structures on  $S$  to the Teichmüller space  $\mathcal{T}(S)$  of  $S$ . This map makes the space  $\mathcal{P}(S)$  a fiber bundle over Teichmüller space.

As it is the case for hyperbolic structures, projective structures, through their holonomy representation, can be studied in the context of the representation theory of the fundamental group of the surface  $S$  in the group  $\mathrm{PSL}(2, \mathbb{C})$ . There is a complex structure on the space  $\mathcal{P}(S)$ , and from works of Hejhal, Earle and Hubbard, it follows that the holonomy map from the space  $\mathcal{P}(S)$  to the character variety of representations of  $\pi_1(S)$  in  $\mathrm{PSL}(2, \mathbb{C})$  is a local biholomorphism. Chapter 12 contains a review of basic properties of holonomy maps of projective structure, as well as a discussion of other issues of representation theory (discreteness, degeneration, etc.) that have been studied in depth by various authors, in particular by D. Dumas.

There are several ways of parameterizing projective structures on surfaces, and one classical way uses Schwarzian derivatives. The Schwarzian derivative is a differential operator which is invariant under Möbius transformations. It was already studied in the nineteenth century, in relation with the *Schwarzian differential equation*  $w''(z) + \frac{1}{2}q(z)w(z) = 0$ , where  $z$  varies in a domain of the Riemann sphere and where  $q$  is a holomorphic function.

We recall that the Schwarzian derivative of a Möbius transformation is zero, and that, in some sense, the Schwarzian derivative of a conformal map is a measure of how far this map is from being a Möbius transformation. The Schwarzian derivative can also be considered as a measure of the difference between two projective structures.

There is an intimate relationship between Schwarzian derivatives and quadratic differentials, the latter being certainly more familiar to Teichmüller theorists. The Schwarzian derivative  $Sf$  of a holomorphic function  $f$  of one complex variable is defined by the formula

$$(Sf)(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2} \left( \frac{f''(z)}{f'(z)} \right)^2$$

(the formula is not important for what follows). A quadratic differential appears from a projective structure on a surface by taking the Schwarzian derivative of a developing map of that structure. Using this fact, Schwarzian derivatives establish a correspondence between projective structures on a given surface with the space of holomorphic quadratic differentials on that surface. In this correspondence, each fiber  $\pi^{-1}(X)$  of the map  $\pi : \mathcal{P}(S) \rightarrow \mathcal{T}(S)$  over a point  $X$  in  $\mathcal{T}(S)$  is identified with the vector space of holomorphic quadratic differentials on a Riemann surface representing  $X$ . Since the vector space of holomorphic quadratic differentials over a surface is also the cotangent space to Teichmüller space at the corresponding point, the theory of the Schwarzian derivative makes an identification between the space  $\mathcal{P}(S)$  and the cotangent bundle  $T^*\mathcal{T}(S)$  of Teichmüller space.

By the Riemann–Roch theorem, the space of quadratic differentials on a closed Riemann surface of genus  $g \geq 2$  is a complex vector space of dimension  $3g - 3$ . As a consequence, the space of projective structures is a fiber bundle over Teichmüller space, with fiber a complex vector space of dimension  $3g - 3$ . This directly shows that  $\mathcal{P}(S)$  is homeomorphic to a cell of complex dimension  $6g - 6$ .

The parametrization of  $\mathcal{P}(S)$  by the fiber bundle of quadratic differentials obtained via the Schwarzian derivative is called the *Schwarzian parametrization* of  $\mathcal{P}(S)$ .

Thurston produced another parametrization for the space  $\mathcal{P}(S)$ , whose definition uses the techniques of hyperbolic geometry and of measured laminations. This is a parametrization by  $\mathcal{ML}(S) \times \mathcal{T}(S)$ , where the Teichmüller space  $\mathcal{T}(S)$  is seen as a space of (equivalence classes) of hyperbolic structures and where  $\mathcal{ML}(S)$  is the space of measured laminations on  $S$ .<sup>9</sup> The homeomorphism  $\mathcal{ML}(S) \times \mathcal{T}(S) \simeq \mathcal{P}(S)$  uses Thurston's general *grafting* operation, which is one of the main tools in the geometric study of complex projective structures. This operation produces from a measured lamination on a hyperbolic surface, considered as a projective structure, a new projective structure. Grafting is first defined when the measured lamination is a weighted simple closed geodesic. In this case, one cuts the surface along that closed geodesic, and introduces between the two boundary components thus obtained a Euclidean annulus whose circumference is equal to the common length of the geodesic boundary components and whose width is determined by the transverse measure of the closed geodesic (seen as an element of  $\mathcal{MF}$ ) that we started with. Grafting a hyperbolic structure over an arbitrary measured lamination  $\mu$  is then defined by taking a sequence of weighted simple closed geodesics converging to  $\mu$  and showing that there is a limiting complex projective structure, which is well defined independently of the approximating sequence. The resulting homeomorphism  $\text{Gr}: \mathcal{ML}(S) \times \mathcal{T}(S) \rightarrow \mathcal{P}(S)$  is called the *grafting homeomorphism*. Continuity, smoothness, properness and other properties of various maps that are associated to the grafting construction were studied by Tanigawa, Scannell, Wolf, Dumas and others, and they are discussed by David Dumas in Chapter 12 of this volume.

Thurston defined a conformal Kobayashi-like distance on each projective surface, which we call the Thurston distance. There is an infinitesimal version of the Thurston distance, in which the norm of a tangent vector  $v$  is the infimum of the norm of all vectors  $v'$  in the Poincaré disk, such that there exists a complex projective immersion of this disk into the surface, sending  $v'$  to  $v$ . This definition is analogous to the definition of the infinitesimal Kobayashi distance on a complex space, where one also takes the infimum over all holomorphic immersions of the Poincaré disk. When the projective surface is obtained by a simple grafting operation (that is, the operation of inserting a Euclidean annulus in a hyperbolic surface), the Thurston metric is the one induced by the length structure associated to the constant-curvature structures on the parts.

Chapter 12 of this volume contains a detailed exposition of the Schwarzian and of the grafting parametrizations of the space  $\mathcal{P}(S)$  of equivalence classes of projective

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<sup>9</sup>Thurston's work on that subject is essentially unpublished, and one proof of the isomorphism  $\mathcal{ML}(S) \times \mathcal{T}(S) \simeq \mathcal{P}(S)$  was written by Y. Kamishima and S. P. Tan.

structures, and a study of the various relations between these two parametrizations. This involves an analysis of the relation between quadratic differentials and grafting as well as a study of the asymptotic aspects of  $\mathcal{P}(S)$  related to the two parametrizations. The chapter also contains a report on fundamental relations between holonomy homomorphisms of projective structures and the grafting construction. It also contains a description of the holonomy representation of a projective structure in terms of convex hulls, pleating loci and the bending deformation in 3-dimensional hyperbolic space. These constructions are at the basis of the beautiful relations between complex projective geometry on surfaces and 3-manifold topology, whose study was started by Sullivan and Thurston and which later on was developed by Epstein & Marden.

Chapter 12 also contains an exposition of results by Dumas on a grafting map compactification of the space  $\mathcal{P}(S)$ , and a description of a fiber of the map  $\mathcal{P}(S) \rightarrow \mathcal{T}(S)$  with respect to this compactification. For a given point  $X$  in Teichmüller space, this description involves a beautiful map  $i_x : \mathbb{P}\mathcal{ML}(S) \rightarrow \mathbb{P}\mathcal{ML}(S)$  called the *antipodal involution*, obtained by transporting the involution  $\phi \mapsto -\phi$  defined on the space  $Q(X)$  of quadratic differentials using the Hubbard–Masur parametrization of  $Q(X)$  by the space of measured foliations  $\mathcal{ML}(S)$  on the surface  $S$ . Dumas also studied another compactification of fibers of  $\mathcal{P}(S) \rightarrow \mathcal{T}$ , which he calls the *Schwarzian compactification*. It is obtained by attaching a copy of the projective space of quadratic differentials of a Riemann surface representing the given point in  $\mathcal{T}(S)$  by taking limits of Schwarzian derivatives. He presents a result that compares the two compactifications of the fibers.

### 3.2 Circle packings

In Chapter 13 of this Handbook, Sadayoshi Kojima reports on rigidity and on flexibility properties of circle packings on complex projective surfaces, and on the relation of circle packings with Teichmüller space.

A *circle* in the complex projective line  $\mathbb{CP}^1 = \mathbb{S}^2$  can be viewed as either a geometric circle for the canonical metric on the sphere  $\mathbb{S}^2$ , or, using the stereographic projection that identifies  $\mathbb{S}^2$  with  $\mathbb{C} \cup \{\infty\}$ , as a Euclidean circle or a straight line in  $\mathbb{C}$ . Circles are invariant by complex projective transformations. As a matter of fact, complex transformations are characterized by the fact that they send circles to circles. This shows that we have a natural local notion of a “circle” on a complex projective surface. In this sense, on a hyperbolic surface, geodesics, horocycles, hypercycles and geometric circles are all circles with respect to the underlying projective structure. This can be clearly seen by taking one of the usual models of hyperbolic space.

In Chapter 13, Kojima studies circles and circle packings on projective Riemann surfaces. Here, the definition of a circle is more restrictive, and one calls *circle* a homotopically trivial simple closed curve that is locally contained in a circle of  $\mathbb{S}^2$ , the term “locally” referring to the image of the curve under the local charts of the projective structure. A *circle packing* is a collection of circles meeting tangentially, with the property that all the complementary regions are curvilinear triangles.

Although interesting problems on circle packings were already noticed by Koebe in the 1930s (and probably before), it is certainly William Thurston who made this into a subject in itself; first in his 1976 Princeton Notes, and then in 1985, when he made the conjecture that certain maps between circle packings converge to conformal maps. This conjecture was proved in 1987 by Burton Rodin and Dennis Sullivan, and it can be considered now as being at the heart of the theory of discrete conformal maps.

Chapter 13 first gives a report on Thurston's reconstruction and generalization of Andreev's theorem on circle packings, following Chapter 13 of Thurston's Princeton 1976 Notes. Andreev's theorem, as revisited by Thurston, is an existence and uniqueness result. The existence part says that a given graph on a Riemann surface determines a constant curvature surface equipped with a circle packing whose combinatorics is encoded by the graph. The uniqueness part says that two such structures encoded by the same graph are related by a global projective map. The only requirement on the graph is that its lift to the universal cover of the surface is a genuine triangulation.

The question of the realization of circle packings on Riemann surfaces was already studied by Koebe in the 1930s. Andreev's work on the subject was published in 1970. The results by Koebe and Andreev concern the case of the closed surface of genus zero (that is, the sphere). Thurston worked out the case of arbitrary genus. In the case of genus  $\geq 2$ , Thurston's result states that there is a unique hyperbolic structure equipped with a circle packing, realizing the given combinatorics.

Kojima then reports on flexibility results whose starting point is a work by R. Brooks who studied, instead of circle packings, more general circle patterns of circles, where complementary regions are allowed to be either triangles or quadrilaterals. Kojima reports on a method due to Brooks of parametrizing these generalized circle patterns by continued fractions, in the case where one of the complementary components is a quadrilateral. The idea is natural, and it consists in trying to fill in the quadrilateral region by successively inserting circles tangent to the rest of the configuration. Adding a new circle creates in general a new quadrilateral, but there are exceptional cases where the added circle is tangent to all the boundary sides of the quadrilateral. In this case the result is a genuine circle pattern, which, as was said before, is a rigid object, and the process ends there. Brooks continued fraction parameter is a projective invariant.

Kojima also outlines recent work on the moduli spaces of pairs  $(S, P)$  where  $S$  is a projective surface and  $P$  a circle packing whose combinatorics is fixed. He describes a projective invariant for such pairs, based on the cross ratio, which was worked out in joint work by Kojima, Mizushima & Tan. The deformation space has a natural structure of a semi-algebraic space. In the last part of Chapter 13, Kojima formulates and motivates a conjecture that states a precise relation between this parameter space and Teichmüller space.

### 3.3 Lorentzian geometry

A few words about Lorentzian geometry are in order.

A Lorentzian  $n$ -manifold  $M$  is a smooth  $n$ -dimensional manifold equipped with a nondegenerate bilinear symmetric form of signature  $(-, +, \dots, +)$  at the tangent space at each point of  $M$ . A Lorentzian  $n$ -manifold is a pseudo-Riemannian manifold of signature  $(1, n - 1)$ . Denoting the bilinear form by  $\langle \cdot, \cdot \rangle$ , if  $v$  is a tangent vector, then the real number  $\langle v, v \rangle$  (and not its square root) is called the *norm* of  $x$ .

Lorentzian manifolds are the most important pseudo-Riemannian manifolds after the Riemannian ones. This is due in part to the use of Lorentzian manifolds in physics. Indeed, 4-dimensional Lorentzian geometry is the setting of general relativity. As a consequence, the language of Lorentzian geometry is often borrowed from the language of physics. For instance, the local parameters in a Lorentzian 4-manifold are seen as three *spatial* parameters and one *temporal* parameter.

From the mathematical point of view, the basic problems of general relativity can be stated in terms of finding Lorentzian metrics on some given manifold that satisfy some partial differential equation (namely, Einstein's equations) involving the Ricci and the scalar curvature tensors.

As in Riemannian geometry, there is a notion of norm-preserving parallel vector transport in Lorentzian geometry. A Lorentzian manifold has a unique affine torsion-free connection which preserves the Lorentzian metric, which is also called the Levi-Civita connection. There are associated notions of curvature, of geodesics and of exponential map. However, the intuition that we have in Riemannian geometry may be misleading in Lorentzian geometry, partly because norms of vectors in a Lorentzian manifold can be negative. One consequence is that in general, geodesics are not distance-minimizing.

We need to recall some more terminology. A tangent vector to a Lorentzian manifold is said to be *time-like* (respectively *space-like*) if its norm is negative (respectively positive). A nonzero vector of zero norm is said to be a *light vector*. A *causal* vector is either a time-like vector or a light vector. A  $C^1$  curve in a Lorentzian manifold is *time-like* (respectively, *space-like*, etc.) if all of its tangent vectors are time-like (respectively, space-like, etc.). A hypersurface in a Lorentzian manifold is *space-like* if the restriction of the Lorentzian metric tensor to the tangent space at each point of that hypersurface is Riemannian. A *flat spacetime* is an oriented Lorentzian manifold together with an orientation for every causal curve. A *Cauchy surface* in a flat spacetime is a codimension-one isometrically immersed Riemannian submanifold which intersects in exactly one point every maximally extended causal curve. A flat spacetime is said to be *globally hyperbolic* if it admits a Cauchy surface. The concept of Cauchy surface was introduced by physicists working in general relativity, and it turned out to be a fundamental concept in Lorentzian geometry, as we shall see below. From the physics point of view, the existence of a Cauchy surface has to do with the so-called "causality condition", which says that there are no time-like closed curves, as it is expected in reality.

Unlike the case of Riemannian manifolds, it is not true that any smooth manifold admits a Lorentzian structure. On the other hand, an important feature of Lorentzian geometry which parallels the Riemannian case is that two Lorentzian manifolds of the same dimension and having the same constant curvature are locally isometric.

Minkowski  $n$ -space, that is, the vector space  $\mathbb{R}^n$  equipped with a nondegenerate symmetric bilinear form of signature  $(-, +, \dots, +)$ , is a linear model for Lorentzian  $n$ -manifolds. The Minkowski model of  $(n - 1)$ -dimensional hyperbolic geometry sits inside Minkowski  $n$ -space as one sheet of a hyperboloid with two sheets. This is a hypersurface that consists of future-directed time-like vectors. In fact, Minkowski Lorentzian  $n$ -space is foliated by  $(n - 1)$ -Riemannian manifolds of constant negative curvature. This should be a hint for a strong relationship between Lorentzian geometry and hyperbolic geometry.

In each dimension  $n$  and for every real number  $\kappa$ , there is a “model Lorentzian manifold”  $X$ , that is, a unique simply connected Lorentzian manifold of dimension  $n$  and of constant curvature  $\kappa$ . Furthermore, such a space  $X$  has the “analytic continuation property”, that is, every isometry between two open sets of  $X$  extends to a global isometry of  $X$ . Using this fact, a Lorentzian manifold of constant curvature can be considered as a homogeneous geometric structure, that is, as a  $(G, X)$  manifold in the sense of Ehresmann. Thus, a Lorentzian manifold of constant curvature can be defined by an atlas whose charts take their values in the model manifold  $X$  and whose coordinate change functions are restrictions of isometries of the model manifold. Again, as in the Riemannian case, there is a notion of developing map and of holonomy representation. Restricting to  $\kappa \in \{0, -1, 1\}$ , the model spaces for 3-dimensional Lorentzian manifolds are called the 3-dimensional Minkowski spacetime ( $\kappa = 0$ ), de Sitter spacetime ( $\kappa = 1$ ), and anti de Sitter spacetime ( $\kappa = -1$ ). De Sitter space can be thought of as the space of planes in hyperbolic space.

We now restrict the discussion to 3-dimensional (more commonly called  $(2 + 1)$ -dimensional) Lorentzian manifolds.

In 1990, Geoffrey Mess wrote a fundamental paper, called *Lorentz spacetimes of constant curvature*.<sup>10</sup> The paper realized a major breakthrough in the field; in particular because it brought into Lorentzian geometry the techniques that had been introduced a few years before by Thurston in hyperbolic geometry and in complex projective geometry (measured laminations, group actions on trees, earthquakes, grafting, bending, and so on).

To say it in very few words, Mess obtained a classification of the space of Lorentzian metrics of constant curvature on manifolds which are of the form  $S \times \mathbb{R}$ , where  $S$  is a closed orientable surface  $S$  of genus  $\geq 2$ . In other words, Mess gave a geometric parametrization of the moduli space of  $(2+1)$  maximal globally hyperbolic spacetimes of constant curvature  $\kappa$ , for  $\kappa \in \{-1, 0, 1\}$ , that contain a compact Cauchy surface. (The case  $\kappa = 0$  was completed by Kevin Scannell in 1999). The problem that Mess solved was explicitly posed by Edward Witten in 1989. As it is the case in Teichmüller theory, there is a natural equivalence relation on the space of Lorentzian metrics of

<sup>10</sup>For 14 years, this paper was circulated as a preprint; it is now published in *Geometriae Dedicata*.



constant curvature, and two metrics on  $S \times \mathbb{R}$  are equivalent if they are isotopic (that is, if they differ by a diffeomorphism of  $S \times \mathbb{R}$  which is isotopic to the identity). Mess showed that the space of equivalence classes of metrics satisfying the above properties is a generalized Teichmüller space parametrized by  $\mathcal{T} \times \mathcal{ML}$ , where  $\mathcal{T}$  is the (usual) Teichmüller space of  $S$  and  $\mathcal{ML}$  its space of measured laminations. It is useful to recall here the following two facts:

- There is a well-known homeomorphism between the space  $\mathcal{T} \times \mathcal{ML}$  and the cotangent bundle of the Teichmüller space of  $S$ , obtained as a consequence of the result by Hubbard and Masur stating that the space of holomorphic quadratic differentials on a Riemann surface (which can be naturally identified with the cotangent space to Teichmüller space at the point represented by that surface) can be identified with the space of measured laminations on that surface.
- The space  $\mathcal{T} \times \mathcal{ML}$  is also reminiscent of the parametrization of the space of equivalence classes of complex projective structures on  $S$ , obtained through Thurston's grafting operation. This is not a pure coincidence, and grafting plays an essential role in this work of Mess.

It is also interesting to note that Mess obtained a new proof of Thurston's earthquake theorem for the case of compact surfaces, using his classification of spacetimes.

Generalizing Mess's work to the case where the surface  $S$  is not compact requires more than the grafting operation. The *canonical Wick rotation*, which has been introduced in this context by Benedetti and Bonsante, is another basic tool for understanding the space of Lorentzian metrics of constant curvature on the product  $S \times \mathbb{R}$ , and explaining the parametrization by  $\mathcal{T} \times \mathcal{ML}$ . The Wick rotation<sup>11</sup> is a transformation, acting as a  $\pi/2$ -rotation, that relates Lorentzian geometry and Riemannian geometry. Roughly speaking, the idea is to consider the parameter  $t$  in the formula  $ds^2 = -dt^2 + dx_1^2 + dx_2^2 + \cdots + dx_{n-1}^2$  defining a Riemannian metric, and the formula  $ds^2 = dt^2 + dx_1^2 + dx_2^2 + \cdots + dx_{n-1}^2$ , defining a Lorentzian metric, as restrictions of one complex parameter to the imaginary axis and to the real axis respectively. The Wick rotation was already successfully used in physics. In particular, it established a relation between the Schrödinger equation of quantum mechanics and the heat equation of thermodynamics.<sup>12</sup>

Given a manifold  $M$  equipped with a Riemannian metric and a non-vanishing vector field  $X$ , the Wick rotation produces a Lorentzian metric on  $M$  for which  $X$  is a timelike vector field. The Lorentzian metric also depends on the choice of two

<sup>11</sup>Named after the Italian theoretical physicist Gian-Carlo Wick (1909–1992).

<sup>12</sup>There are several well-known occurrences in geometry where the fact of complexifying a real parameter turns out to be very fruitful. To stay close to our subject matter, we can just mention here the complexification of earthquake coordinates which establishes relations between Weil–Petersson geometry, projective structures, pleated surfaces and quasifuchsian groups (see e.g. the work of McMullen on the extension of earthquake paths to proper holomorphic maps from disks into Teichmüller space), the complexification of Thurston's shear coordinates for measured laminations which also gives a parametrization of the space of hyperbolic 3-manifolds with fundamental group equal to a surface fundamental group (work of Bonahon), or the complex measures that define quake-bend maps that appear in the work of Epstein and Marden, where real measures correspond to earthquakes and imaginary measures correspond to bending.

positive functions  $\alpha$  and  $\beta$ , called *rescaling functions*. The notion of rescaling is another fundamental object in the theory that is developed by Riccardo Benedetti and Francesco Bonsante in Chapter 14 of this volume. Another important tool in this theory is the notion of *cosmological time*, introduced in this context by Benedetti and Guadagnini.

Finally, we mention that the Wick rotation-rescaling theory also provides geometric relations between spacetimes of different curvatures, and between such spacetimes and complex projective structures. The theory transforms the various spacetimes into hyperbolic 3-manifolds that carry at infinity the same projective structure.

### 3.4 Fricke–Klein coordinates

As is well known, the Teichmüller space of a surface  $S$  can be described as a subspace of a space of conjugacy classes of representations of the fundamental group of  $S$  in Lie groups, in particular the Lie group  $SL(2, \mathbb{C})$ . This point of view was already used by R. Fricke and F. Klein in the nineteenth century. It is a well-known fact that the trace of a  $2 \times 2$  matrix is a conjugacy invariant, and Fricke and Klein studied the question of parametrizing spaces of conjugacy classes of representations of the fundamental group of a surface in  $SL(2, \mathbb{C})$  by a finite number of traces, viz. traces of images of base elements of the group and of some of their combinations. The space of conjugacy classes of representations is referred to here as the *character variety*. Trace coordinates are often called *Fricke–Klein coordinates*. In the case where the fundamental group of the surface is a free group of rank two, a result of Vogt, Fricke and Klein, which is quoted in several chapters of this volume, gives a characterization of two-variable functions that are invariant under the action of  $SL(2, \mathbb{C})$  on itself by conjugation. This characterization leads to a description of the character variety by a set of polynomial equations, involving the traces of the images of three elements of the fundamental group.

Chapter 15 of this volume, written by Bill Goldman, considers Fricke–Klein coordinates in detail. Goldman presents the complete results with explicit formulae in the case of two- and three-generator surface groups. Non-orientable surfaces are also considered. The chapter also contains an exposition of the background material in invariant theory and in hyperbolic geometry that is needed in order to obtain the formulae. Goldman also gives formulae relating the trace coordinates to the Fenchel–Nielsen coordinates in the case of a particular two-generator surface, namely the one-holed torus.

### 3.5 Diagrammatic approach

Chapter 16 by Sean Lawton and Elisha Peterson concerns the character variety of  $SL(2, \mathbb{C})$ -representations of the free group  $F_2$  on two generators. One obvious relation with surface geometry stems from the fact that  $F_2$  is the fundamental group of the pair

of pants and of the torus with one hole. The term *character variety* refers here to the orbit space of the subset of completely reducible representations under the action of  $SL(2, \mathbb{C})$  by conjugation. As is well known, this character variety is an algebraic set. It contains the Teichmüller space of the surface as a subspace, and it also contains moduli spaces of other geometric structures.

The main object of this chapter is to develop a diagrammatic approach to the study of the character variety. The diagrams that appear here are graphs called *spin networks*. These graphs are used as a diagrammatic tool in the description of a natural additive basis for the coordinate ring of the character variety. The elements of this basis are the *central functions*, and the authors make a detailed study of the properties of this basis. Diagrammatic calculus is used to make explicit the symmetries of this basis. The authors also give a new constructive proof of results by Vogt and Fricke–Klein that are considered from a different viewpoint in Chapter 15 by Goldman.

Diagrammatic calculus has been thoroughly used by physicists, the most notable examples being certainly the diagrams that appear in the works of Richard Feynman and of Roger Penrose. In mathematics, it is known that diagrammatic calculus considerably simplifies certain proofs and algebraic computations.

Spin networks, as a diagrammatic tool, have been previously used in the description of quantum angular momentum by Penrose. They also appear, together with central function bases, in the work of John Baez (1996) in relation to gauge theory. More recent related work was done by Adam Sikora (2001), who considered graphs similar to spin networks, and who used the graphical calculus in the deformation theory of the  $SL_3$ -character variety of the fundamental group of a 3-manifold, with a view on applications to quantum invariants of 3-manifolds. We finally mention that Florentino, Mourão and Nunes (2004) used similar tools in a work that is related to the geometric quantization of the moduli space of flat connections on a Riemann surface.

## 4 The Grothendieck–Teichmüller theory

The *Grothendieck–Teichmüller theory* is an expression that was coined after Alexandre Grothendieck wrote his *Esquisse d'un programme* (1983), a detailed research program which was part of an application for a researcher position at CNRS. The theory that is referred to in this expression has several facets, and the Grothendieck–Teichmüller theory that is reported on in this volume includes the subjects of *dessins d'enfants*, the *reconstruction principle*, and the theory of the *solenoid*.

Let me start by saying a few words on some of the objects that play important roles in this theory, namely, *dessins d'enfants*, the absolute Galois group, towers, profinite groups and the Grothendieck–Teichmüller modular group.

A *dessin d'enfant* is a finite graph embedded in an oriented connected surface, which has the following two properties:

- the complement of the graph is a union of cells;

- the vertices of the graph are colored black or white in such a way that the endpoints of any vertex do not have the same color.

One may wonder how such a simple definition leads to important developments, but in some sense this is often the case in mathematics.

It is good to recall that important ideas in Grothendieck–Teichmüller theory originate in algebraic geometry.

Grothendieck introduced *dessins d'enfants* in 1984 as a tool for the study of the absolute Galois group of the field of rational numbers, and in relation with some holomorphic branched covers of surfaces called *Belyi functions*. After that, the use of *dessins d'enfants* in Riemann surface theory and in low-dimensional topology has been highlighted by many authors. It turned out that *dessins d'enfants* make connections between several fields of mathematics, e.g. the Galois theory of algebraic numbers, Riemann surfaces, combinatorial group theory and hyperbolic geometry.

Let us note that *dessins d'enfants* were already used in two chapters of Volume I of this Handbook, namely, those written by Harvey and by Herrlich & Schmithüsen, in relation with Teichmüller disks. It is sometimes useful to have different points of view on an important topic, written by different authors. In Chapter 18 of the present volume, *dessins d'enfants* are considered in more detail, and from a point of view closer to that of Grothendieck's original. This point of view heavily uses the language and techniques of algebraic geometry. *Dessins d'enfants* are also considered, in the same chapter, in relation with *origamis*, which are special classes of Riemann surfaces on which significant progress has been made recently.

As already stated, the Grothendieck–Teichmüller theory studies actions of the absolute Galois group  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  of the field of rational numbers. Here,  $\overline{\mathbb{Q}}$  is the field of algebraic numbers, that is, the algebraic closure of the field  $\mathbb{Q}$  of rational numbers, and  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  is the topological automorphism group of the Galois extension  $\overline{\mathbb{Q}}/\mathbb{Q}$ . We note that there is no explicit description of the absolute Galois group  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ , and partial understanding of this group is obtained by studying its actions on various spaces.

It is also worth noting that the representation theory of the absolute Galois group plays an important role in Wiles' proof of Fermat's Last Theorem.

One relation of Grothendieck's work with Teichmüller theory stems from the fact that one of Grothendieck's approaches to the analysis of the group  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  is via the action of that group on the “system” of all moduli spaces  $\mathcal{M}_{g,n}$  (for varying  $g$  and  $n$ ). Grothendieck calls this system the *Teichmüller tower*. In practice, a tower in this context is an object obtained either as the inverse limit of spaces, or as a profinite completion of groups. The word “tower” occurs at several places in the Grothendieck–Teichmüller theory. For instance, one has “towers of surfaces”, “towers of Teichmüller spaces”, “towers of fundamental groups”, “towers of mapping class groups” and so on. The Grothendieck–Teichmüller theory studies automorphisms of these objects, and makes relations between these objects and actions of the Galois groups on various associated spaces.

As already mentioned, the notion of profinite group is an important object in this theory. We recall that a profinite group is a Hausdorff, compact and totally disconnected

topological group which is isomorphic to a projective limit of an inverse system of finite groups. In some sense, a profinite group is obtained by assembling finite groups, and hence, profinite groups may be understood by studying their finite quotients. The absolute Galois group  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  is itself an example of a profinite group. Indeed,  $\overline{\mathbb{Q}}$  is the union of all the Galois finite normal extensions of  $\mathbb{Q}$  in  $\mathbb{C}$ , and  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  is a projective limit of the finite Galois groups of these extensions. Algebraic fundamental groups of schemes, that appear in algebraic geometry, are other examples of profinite groups. (But fundamental groups in the sense of algebraic topology are not.) Any group  $G$  has a profinite completion  $\hat{G}$ , defined as the projective limit of the groups  $G/N$ , where  $N$  varies over the finite-index normal subgroups of  $G$ . There is a natural homomorphism  $G \rightarrow \hat{G}$ , which satisfies a natural universal property, and the image of  $G$  under this homomorphism is dense in  $\hat{G}$ .

The *Grothendieck–Teichmüller modular group* has been defined by Drinfel'd in 1991, as an extension of the absolute Galois group  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ . This result by Drinfel'd somehow gave a natural setting for the action of the Galois group on the Teichmüller tower that was alluded to by Grothendieck in his *Esquisse d'un programme*. The Grothendieck–Teichmüller modular group is also the automorphism group of a tower of fundamental groupoids of a stack of moduli spaces equipped with *tangential base-points*. This group was studied by L. Schneps, P. Lochak, H. Nakamura, H. Tsunogai, H. Voelklein and T. Shaska and others. L. Schneps identified the Grothendieck–Teichmüller modular group with the automorphism group a tower of profinite completions of Artin braid groups. Let us also mention that the Grothendieck–Teichmüller theory has also applications in conformal field theory, and that there is a work in this direction done by B. Bakalov and A. Kirillov (related to previous work of Moore and Seiberg). More recently, P. Hu & I. Kriz worked out new relations between the Grothendieck–Teichmüller theory and conformal field theory. They described actions of the Galois group of a number field on the category of modular functors. We shall see in Chapter 18 that the Galois group of  $\mathbb{Q}$  also acts on origamis, which are closely related to dessins d'enfants.

The reconstruction principle is another important aspect of Teichmüller theory that was formulated by Grothendieck, inspired from ideas that originate in algebraic geometry. Chapter 17 of the present volume contains a detailed overview on that theory, written by Feng Luo, with an exposition of several important applications of that principle in low-dimensional topology. The reconstruction principle is related to the study of the Teichmüller tower and it gives rise to new kind of geometric structures, namely,  $(\mathbb{Q}P^1, \text{SL}(2, \mathbb{Z}))$  structures, also called *modular structures*.

I have included the chapter on the Teichmüller space of the solenoid in the part of this volume dedicated to the Grothendieck–Teichmüller theory, because the study of the solenoid involves the Teichmüller tower, the mapping class group tower and other similar objects whose study is inherent in Grothendieck's program, without the language of algebraic geometry. This chapter could also have been included in Part A on the metric and the analytic theory, but I have the feeling that the fact of including it in the part on the Grothendieck–Teichmüller theory opens up a nice perspective.

Now let us review in more detail the three chapters that constitute Part D of this volume.

#### 4.1 The reconstruction principle

Let  $S$  be a compact surface of negative Euler characteristic. An *essential subsurface*  $S'$  of  $S$  is a surface with boundary and with negative Euler characteristic embedded in  $S$ , such that no boundary component of  $S'$  bounds a disk in  $S$  or is isotopic to a boundary component of  $S$ . There is a hierarchy on the set of essential subsurfaces of  $S$ , in which the *level* of a surface  $S'$  is the maximal number of disjoint simple closed curves that cuts it into pairs of pants. In particular, level-zero surfaces are the pairs of pants, level-one surfaces are the four-punctured spheres and the one-holed tori, and level-two surfaces are the two-holed tori and the five-holed spheres.

Grothendieck's reconstruction principle says that some of the most important geometric, algebraic and topological objects that are associated to a surface  $S$  (e.g. the Teichmüller space, the mapping class group, the space of measured foliations, and spaces of representations in  $SL(2, K)$  for a given field  $K$ ) can be reconstructed from the corresponding spaces associated to the (generally infinite) set of level-zero, level-one and level-two essential subsurfaces of  $S$ . The geometric structures on the level-zero spaces are the building blocks of the general structures, and the structures on the level-one and the level-two spaces are the objects that encode the gluing. Paraphrasing Grothendieck from his *Esquisse d'un programme*, "the Teichmüller tower can be reconstructed from level zero to level two, and in this reconstruction, level-one gives a complete set of generators and level-two gives a complete set of relations".

Grothendieck's ideas were inspired by analogous situations in algebraic geometry, in particular by ideas originating in reductive group theory, where the semi-simple rank of a reductive group plays the role of "level".

In a series of extremely interesting and original papers, Feng Luo developed Grothendieck's intuition and made it precise. Chapter 17 of this volume, written by Luo, constitutes a detailed survey of various results in this theory.

A fundamental new object that appears in this theory is the notion of *modular structure*, a  $(\mathbb{Q}P^1, SL(2, \mathbb{Z}))$  structure in the usual sense of a geometric structure defined by an atlas. Here,  $\mathbb{Q}P^1 = \mathbb{Q} \cup \{\infty\}$  is seen as the set of rational points on the unit circle. Luo shows that the set of isotopy classes of essential simple closed curves on an oriented surface of level at least one is equipped with a modular structure which is invariant under the action of the mapping class group of the surface. The atlas for such a structure is obtained through some coordinate charts associated to level-one essential subsurfaces of the original surface. For these level-one surfaces, coordinate charts are homeomorphisms onto  $\mathbb{Q}P^1$ . We note that the idea of a modular structure for the set of isotopy classes of essential simple closed curves on the four-punctured sphere is already inherent in the work of Max Dehn done in the 1930s. It is easy to see that there is also a modular structure on the space of essential curves on the torus, and that this structure is natural with respect to the action of the mapping class group

of the torus on the space of curves. Luo describes in Chapter 17 a modular structure on the set of isotopy classes of pair of pants decompositions of a surface.

Another application of the reconstruction principle presented in Chapter 17 concerns characters of  $\mathrm{SL}(2, K)$ -representations, where  $K$  is an arbitrary field. Let us review the definition.

Let  $\mathcal{S}(S)$  be the set of isotopy classes of essential simple closed curves on  $S$ . An  $\mathrm{SL}(2, K)$ -character on  $\mathcal{S}(S)$  is defined here as the map induced by the trace function of a representation of  $\pi_1(S)$  in  $\mathrm{SL}(2, K)$ . Luo calls an  $\mathrm{SL}(2, K)$ -trace function on  $\mathcal{S}(S)$  a function  $\mathcal{S}(S) \rightarrow K$  whose restriction to every subset  $\mathcal{S}(S')$  of  $\mathcal{S}(S)$  is an  $\mathrm{SL}(2, K)$ -character on  $\mathcal{S}(S')$ , where  $S' \subset S$  is an essential level-one surface.

From the work of Fricke and Klein to which we already referred at several occasions, it follows that the trace function defined on the fundamental group  $\pi_1(S)$  of  $S$ , with respect to an  $\mathrm{SL}(2, K)$ -representation of  $\pi_1(S)$ , is determined by the restriction of this function to the elements of  $\pi_1(S)$  that are represented by simple curves.

Luo proves that any  $\mathrm{SL}(2, K)$ -trace function on  $\mathcal{S}(S)$  is the  $\mathrm{SL}(2, K)$ -character on that set, except for a finite number of cases which he enumerates. The result was conjectured by Grothendieck. To prove this fact, Luo produces a complete set of equations that express the fact that a function  $\mathcal{S}(S) \rightarrow K$  is an  $\mathrm{SL}(2, K)$ -character, and he proves that these equations are supported on the essential level-two subsurfaces of  $S$ . The consequence is that the character functions satisfy Grothendieck's reconstruction principle, except for a finite number of functions supported on surfaces of genus 0 with  $n \geq 5$  punctures.

Another application of Grothendieck's reconstruction principle, which is also due to Luo, concerns geometric intersection functions defined on  $\mathcal{S}(S)$ . Luo calls a function  $f: \mathcal{S}(S) \rightarrow \mathbb{R}$  a *geometric intersection function* if there exists a measured lamination  $\mu$  on  $S$  such that  $f$  is the intersection function with  $\mu$ , that is,  $f(\alpha) = i(\alpha, \mu)$  for every  $\alpha$  in  $\mathcal{S}$ . Luo proves that a function  $\mathcal{S}(S) \rightarrow \mathbb{R}$  is a geometric intersection function if for every essential level-one subsurface  $S'$  of  $S$ , the restriction of  $f$  to  $\mathcal{S}(S')$  is a geometric intersection function.

A related result, again due to Luo, is that geometric intersection functions on the set of isotopy classes of essential curves of a level-one surface are characterized by two homogeneous equations in the  $(\mathbb{Q}P^1, \mathrm{PSL}(2, \mathbb{Z}))$ -structure on these subsurfaces.

Applications of the reconstruction principles in the study of Teichmüller spaces, measured foliation spaces, and mapping class groups are also discussed in the same chapter.

## 4.2 Dessins d'enfants

In Chapter 18, Frank Herrlich and Gabriela Schmithüsen give an overview of the theory of dessins d'enfants, and of another class of combinatorial objects, namely origamis, and they develop the relation between the two classes.

We already recalled the classical result of Riemann stating that any compact Riemann surface can be defined as an algebraic curve, that is, as the zero set of a two-

variable polynomial. In this setting, the most useful polynomials are probably those whose coefficients are in the field  $\overline{\mathbb{Q}}$  of algebraic numbers. A celebrated result due to G. Belyĭ asserts that any compact Riemann surface represented by an algebraic curve with coefficients in  $\overline{\mathbb{Q}}$  is a ramified meromorphic covering of the Riemann sphere, in which the ramification occurs over at most three points. This leads to the introduction of the following important notion: a *Belyĭ morphism*  $X \rightarrow \mathbb{P}^1(\mathbb{C})$  is a ramified covering from a Riemann surface  $X$  to the complex projective line  $\mathbb{P}^1(\mathbb{C})$ , which is ramified over at most three points. Using this notion, Belyĭ's theorem states that the Riemann surface  $X$  can be defined as an algebraic curve over the field  $\overline{\mathbb{Q}}$  if and only if there exists a Belyĭ morphism  $X \rightarrow \mathbb{P}^1(\mathbb{C})$ .

Chapter 18 contains an exposition of the fact that the following categories of objects are in natural one-to-one correspondence:

- equivalence classes of Belyĭ morphisms;
- equivalence classes of dessins d'enfants;
- equivalence classes of bipartite ribbon graphs;
- conjugacy classes of finite index subgroups of  $F_2$ , the free group on two generators;
- transitive actions of  $F_2$  on a symmetric group  $S_d$  of permutations of  $d$  objects up to conjugacy in  $S_d$ .

From the correspondence between the first two items in this list, it follows that the absolute Galois group  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  acts on the class of equivalence classes of dessins d'enfants. There is still no explicit description of this action, but the correspondence leads to important results, such as the embedding of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  into the Grothendieck–Teichmüller group  $\widehat{GT}$ . We also note that L. Schneps described a faithful action of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  on a class of equivalence classes of trees.

Herrlich and Schmithüsen provide a proof of the fact that the action of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  on dessins d'enfants is faithful. They address the question of finding invariants of the actions mentioned above. In other words, the question is to find properties of equivalence classes of dessins d'enfants (and of the other related objects) that remain invariant under the action of the Galois group. There is no complete list of such invariants, but Herrlich and Schmithüsen study a few invariants such as the *genus* and the *valency lists* of a dessin d'enfants. They explain how the Galois action on dessins induces an injective group homomorphism of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  in the automorphism group  $\text{Aut}(\hat{F}_2)$  of the profinite completion  $\hat{F}_2$  of  $F_2$ . This is then explained in the general context of actions of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  on algebraic fundamental groups of schemes.

The embedding  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{Aut}(\hat{F}_2)$  leads to the introduction of the Grothendieck–Teichmüller group  $\widehat{GT}$ , introduced by Drinfel'd, which is a subgroup of  $\text{Aut}(\hat{F}_2)$  which contains the image of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ .

The second part of Chapter 18 deals with origamis. These are surfaces obtained by taking a finite number of isometric squares in the Euclidean plane and gluing them along their boundaries by using Euclidean translations. Origamis already appeared in Volume I of this Handbook, and they are also mentioned in Chapter 10 of the present



volume, in particular regarding their affine groups. There are several questions on origamis that are still unsolved, regarding their arithmetic theory, their occurrence as Teichmüller disks in moduli space, and so on.

In Chapter 18, origamis are studied in parallel with dessins d'enfants. One can see the relation between these two classes of objects in the following manner: whereas a dessin d'enfant is associated to a finite unramified covering of the sphere with three points deleted, an origami is associated to an unramified finite covering of the torus with one point deleted. It may be useful to note here that the sphere with three punctures and the torus with one puncture are exactly the surfaces whose fundamental group is a free group on two generators.

Herrlich and Schmithüsen give a list of classes of objects that are equivalent to origami curves. This list is analogous to the list that we mentioned above, concerning dessins d'enfants. Then the authors report on the relation between origami curves and dessins d'enfants. More precisely, they show that an origami curve can be interpreted as a dessin d'enfants, and they show by examples how to produce a dessin associated to an origami curve. Dessins d'enfants can also be associated to a *cusp* of an origami curve, that is, a boundary point of the closure of the image of the origami curve in the Deligne–Mumford compactification of moduli space. Herrlich and Schmithüsen also study the action of the absolute Galois group on the set of origamis.

### 4.3 The solenoid

Taking a covering of a Riemann surface leads to a natural operation at the level of Teichmüller spaces. In fact, there is a contravariant functor from the category of oriented closed surfaces, with finite-degree orientation-preserving covers between them as morphisms, to the category of finite-dimensional complex manifolds with holomorphic embeddings as morphisms. This functor associates to each Riemann surface its Teichmüller space and to each orientation-preserving covering  $X \rightarrow Y$ , the naturally induced holomorphic map  $\mathcal{T}(Y) \rightarrow \mathcal{T}(X)$  between the corresponding Teichmüller spaces obtained by lifting conformal structures on  $Y$  to conformal structures on  $X$ . In some sense, the solenoid can be considered as a universal object arising from this theory of taking covers of surfaces.

The solenoid was introduced by Dennis Sullivan in the early 1990s, as the inverse limit of a tower of finite sheeted pointed covers of a pointed closed oriented surface of genus  $g \geq 2$ . In this setting, “pointed” means equipped with a basepoint, all covers are unbranched, and the order relation between pointed covers is defined by the existence of a factorizing cover. We note that the fact of specifying basepoints make factorizations unique whenever they exist.

More precisely, the family  $\mathcal{C}$  of pointed finite-order covers of a pointed base surface  $(S_0, x_0)$ , equipped with the partial order  $\preceq$  defined by factorizations of covers, is inverse directed, and the *compact solenoid* (also called the *universal hyperbolic solenoid*)  $\mathcal{S}$  is the inverse limit of this family. Thus, a point in the compact solenoid  $\mathcal{S}$  is a point  $y_0$  on the base surface  $S_0$  together with a point  $y_i$  on each finite covering

surface  $\pi_i: S_i \rightarrow S_0$  such that  $\pi_i(y_i) = y_0$ , with the property that if two covers  $\pi_i: S_i \rightarrow S_0$  and  $\pi_j: S_j \rightarrow S_0$  satisfy  $\pi_i \leq \pi_j$  and if  $\pi_{i,j}$  is the factorizing covering, then  $\pi_{i,j}(y_j) = y_i$ .

The compact solenoid  $\mathcal{C}$  does not depend on the choice of the base surface  $(S_0, x_0)$ . This is a consequence of the fact that any two finite covers have a common finite cover.

The compact solenoid  $\mathcal{S}$  is equipped with the subspace topology induced from the product topology on the infinite product of all pointed closed surfaces that finitely cover the base surface. With this topology,  $\mathcal{S}$  is compact, and its local structure is that of a surface times a Cantor set. Thus, the compact solenoid has the structure of a *foliated space*, or a *lamination*. (These are spaces more general than the familiar foliated manifolds and laminations on manifolds.) The direction of the Cantor set is called the *transversal direction*. Using the language of foliation theory, the path-connected components of  $\mathcal{S}$  are called the *leaves*. In the solenoid, each leaf is homeomorphic to a disk and is dense in  $\mathcal{S}$ .

Sullivan introduced the compact solenoid as a sort of “universal dynamical system”. Independently of Sullivan’s original motivation, the compact solenoid turned out to be an interesting object that can be studied for itself. Such a study has been carried out by Sullivan, Biswas and Nag, and, more recently, by Šarić, Markovic, Penner and others.

Using the correspondence between unbranched covers of a surface and subgroups of its fundamental group, there is an equivalent definition of the solenoid that uses the directed set of subgroups of the fundamental group of the base surface, equipped with the inclusion order relation.

The compact solenoid can also be described as a principal  $G$ -bundle over the base surface, with  $G$  being the profinite completion of the fundamental group of the surface and with fibers homeomorphic to a Cantor set. In this respect, recall that the universal cover of a pointed surface  $(S_0, x_0)$  is a principal  $\pi_1(S_0, x_0)$ -bundle over that surface, and that the compact solenoid appears as the principal  $G$ -bundle obtained by extending the structure group of this bundle from the fundamental group to its completion. (We recall that any group is naturally included in its profinite completion.) From this description, the compact solenoid can be thought of as a “universal closed surface”. The compact solenoid  $\mathcal{S}$ , as a lamination, has an invariant transverse measure which is induced by the Haar measure on the fiber group. This transverse measure on the solenoid is important. For instance, it can be used for obtaining a measure on the solenoid by taking the product of this transverse measure with the area form obtained from a hyperbolic structure on the leaves. It can also be used for integrating objects like quadratic differentials which are holomorphic on the leaves, and so on.

The compact solenoid is equipped with a rich variety of natural structures, that parallel analogous structures associated to compact surfaces. The examples of such structures that are of main interest for us here are complex structures and hyperbolic structures, and there is a uniformization theorem that connects them. A complex structure on  $\mathcal{S}$  is defined by an atlas whose transition maps are holomorphic when restricted to the local leaves, and are continuous in the transverse directions. The

solenoid, equipped with a complex structure, becomes a *Riemann surface lamination*. There is a notion of a quasiconformal map between Riemann surface laminations. Markovic and Šarić proved that any two homotopic quasiconformal maps between complex solenoids are isotopic by a uniformly quasiconformal isotopy. There is a space of Beltrami differentials on the compact solenoid, and a corresponding Teichmüller space  $\mathcal{T}(\mathcal{S})$ . The latter can be defined, as in the case of the Teichmüller space of a surface, either as a space of equivalence classes of Beltrami differentials, or as a space of equivalence classes of marked solenoids equipped with complex structures. The space  $\mathcal{T}(\mathcal{S})$  is infinite-dimensional and separable (in contrast with infinite-dimensional Teichmüller spaces of surfaces, which are all non-separable). The space  $\mathcal{T}(\mathcal{S})$  can also be naturally embedded as a complex submanifold of the universal Teichmüller space.

Let us mention that there is another object which has the same flavour as the Teichmüller space of the compact solenoid, and which was studied by Biswas, Nag & Sullivan. It is also related to the functor that we mentioned above, between the categories {closed oriented surfaces, finite covers} and {complex spaces, holomorphic maps}. This functor leads to a directed system of Teichmüller spaces, with order relation stemming from existence of holomorphic maps induced from coverings. The direct limit of this system is called the *universal commensurability Teichmüller space*, and it is denoted by  $\mathcal{T}_\infty$ . Like the solenoid itself, the space  $\mathcal{T}_\infty$  does not depend on the choice of the base surface, and it is equipped with a Teichmüller metric, induced from the Teichmüller metrics of the Teichmüller spaces of the surfaces that were used to define it. The space  $\mathcal{T}_\infty$  is also equipped with a Weil–Petersson metric. By a result of Biswas, Nag & Sullivan, the Teichmüller space of the compact solenoid,  $\mathcal{T}(\mathcal{S})$ , is the completion of the universal commensurability Teichmüller space  $\mathcal{T}_\infty$ , with respect to the Teichmüller metric.

We also mention a relation with algebraic geometry. Biswas, Nag & Sullivan used their work on the universal commensurability Teichmüller space to obtain a genus-independent version of determinant line bundles and of connecting Mumford isomorphisms. This theory provides a natural Mumford isomorphism between genus-independent line bundles, which is defined over the universal commensurability Teichmüller space  $\mathcal{T}_\infty$ , made out of the Mumford isomorphisms between determinant line bundles defined at the finite-dimensional levels.

Now back to the compact solenoid.

There is a natural notion of an automorphism group of the Teichmüller space of the solenoid that was also introduced by Biswas, Nag & Sullivan. These authors proved in 1996 that this group is isomorphic to the virtual automorphism group of the fundamental group of the base surface. We recall that the virtual automorphism group of a group  $G$  is the set of isomorphisms between finite index subgroups of  $G$  up to the equivalence relation that identifies two such isomorphisms if they agree on a finite index subgroup. The virtual automorphism group of  $G$  is also called the *abstract commensurator group* of  $G$ . For instance, the virtual automorphism group of  $\mathbb{Z}$  is the multiplicative group  $\mathbb{Q}^*$ . The relation with the solenoid stems from the fact

that there is a natural correspondence between homotopy classes of homeomorphisms between finite covers of a surface and elements of the virtual automorphism group of the fundamental group of that surface. A related natural object of study is the *baseleaf preserving mapping class group* of the compact solenoid  $\mathcal{S}$ , defined (modulo some technicalities) after the choice of a baseleaf, as the group of isotopy classes of baseleaf preserving self-homeomorphisms of this space  $\mathcal{S}$ . C. Odden proved in 2004 that the baseleaf preserving mapping class group of  $\mathcal{S}$  is naturally isomorphic to the virtual automorphism group of the fundamental group of the base surface. This result is considered as an analogue of the Dehn–Nielsen–Baer Theorem that describes the mapping class group of a closed surface of genus  $\geq 1$  as the outer automorphism group of its fundamental group. Markovic & Šarić proved that the baseleaf preserving mapping class group of the solenoid does not act discretely on  $\mathcal{T}(\mathcal{S})$ , a result which should be compared to the fact that in general, the mapping class group of surfaces of infinite type does not act discretely on the corresponding Teichmüller space.

The *non-compact solenoid*, also called the *punctured solenoid*, and denoted by  $\mathcal{S}_{\text{nc}}$ , is defined in analogy with the compact solenoid, as the inverse limit of the system of all pointed finite sheeted coverings of a base surface  $S_0$  of negative Euler characteristic, except that here,  $S_0$  is a punctured surface. A study of the noncompact solenoid was done by Penner & Šarić, who equipped that space with the various kinds of structures that exist on the compact solenoid, namely, complex structures, quasiconformal maps between them, a Teichmüller space, and a mapping class group which is isomorphic to a subgroup of the commensurator group of the base surface preserving the peripheral structure (in analogy with the case of the mapping class group of a punctured surface).

Chapter 19 of this Handbook, written by Dragomir Šarić, contains a review of the theory of the compact solenoid and of recent work on the noncompact solenoid  $\mathcal{S}_{\text{nc}}$  by Penner & Šarić, as well as work by Bonnot, Penner and Šarić on a cellular action of the mapping class group of  $\mathcal{S}_{\text{nc}}$ . In analogy with the corresponding situation for punctured surfaces, there is a decorated Teichmüller space of the noncompact solenoid, with associated  $\lambda$ -length coordinates, and a convex hull construction of fundamental domains which gives an interesting combinatorial structure for this Teichmüller space, generalizing an analogous structure that was developed by Penner for the Teichmüller space of a punctured surface. An explicit set of generators for the mapping class group of the noncompact solenoid is also discussed. Note that no such explicit set of generators for the compact solenoid is known. It is conjectured that the mapping class groups of the compact and of the noncompact solenoids are not finitely generated.

Chapter 19 ends with a discussion of open problems on the Teichmüller space and on the mapping class group of the compact and the noncompact solenoids.