Preface

Algebraic topology is the interplay between "continuous" and "discrete" mathematics. Continuous mathematics is formulated in its general form in the language of topological spaces and continuous maps. Discrete mathematics is used to express the concepts of algebra and combinatorics. In mathematical language: we use the real numbers to conceptualize continuous forms and we model these forms with the use of the integers. For example, our intuitive idea of time supposes a continuous process without gaps, an unceasing succession of moments. But in practice we use discrete models, machines or natural processes which we define to be periodic. Likewise we conceive of a space as a continuum but we model that space as a set of discrete forms. Thus the essence of time and space is of a topological nature but algebraic topology allows their realizations to be of an algebraic nature.

Classical algebraic topology consists in the construction and use of functors from some category of topological spaces into an algebraic category, say of groups. But one can also postulate that global qualitative geometry is itself of an algebraic nature. Consequently there are two important view points from which one can study algebraic topology: homology and homotopy.

Homology, invented by Henri Poincaré, is without doubt one of the most ingenious and influential inventions in mathematics. The basic idea of homology is that we start with a geometric object (a space) which is given by combinatorial data (a simplicial complex). Then the linear algebra and boundary relations determined by these data are used to produce homology groups.

In this book, the chapters on singular homology, homology, homological algebra and cellular homology constitute an introduction to homology theory (construction, axiomatic analysis, classical applications). The chapters require a parallel reading – this indicates the complexity of the material which does not have a simple intuitive explanation. If one knows or accepts some results about manifolds, one should read the construction of bordism homology. It appears in the final chapter but offers a simple explanation of the idea of homology.

The second aspect of algebraic topology, homotopy theory, begins again with the construction of functors from topology to algebra. But this approach is important from another view point. Homotopy theory shows that the category of topological spaces has itself a kind of (hidden) algebraic structure. This becomes immediately clear in the introductory chapters on the fundamental group and covering space theory. The study of algebraic topology is often begun with these topics. The notions of fibration and cofibration, which are at first sight of a technical nature, are used to indicate that an arbitrary continuous map has something like a kernel and a cokernel – the beginning of the internal algebraic structure of topology. (The chapter on homotopy groups, which is essential to this book, should also be studied for its applications beyond our present study.) In the ensuing chapter on duality the analogy to algebra becomes clearer: For a suitable class of spaces there exists a duality theory which resembles formally the duality between a vector space and its dual space.

The first main theorem of algebraic topology is the Brouwer–Hopf degree theorem. We prove this theorem by elementary methods from homotopy theory. It is a fairly direct consequence of the Blakers–Massey excision theorem for which we present the elementary proof of Dieter Puppe. Later we indicate proofs of the degree theorem based on homology and then on differential topology. It is absolutely essential to understand this theorem from these three view points. The theorem says that the set of self-maps of a positive dimensional sphere under the homotopy relation has the structure of a (homotopically defined) ring – and this ring is the ring of integers.

The second part of the book develops further theoretical concepts (like cohomology) and presents more advanced applications to manifolds, bundles, homotopy theory, characteristic classes and bordism theory. The reader is strongly urged to read the introduction to each of the chapters in order to obtain more coherent information about the contents of the book.

Words in boldface italic are defined at the place where they appear even if there is no indication of a formal definition. In addition, there is a list of standard or global symbols. The problem sections contain exercises, examples, counter-examples and further results, and also sometimes ask the reader to extend concepts in further detail. It is not assumed that all of the problems will be completely worked out, but it is strongly recommended that they all be read. Also, the reader will find some familiarity with the full bibliography, not just the references cited in the text, to be crucial for further studies. More background material about spaces and manifolds may, at least for a while, be obtained from the author's home page.

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