

Contents

Preface	v
1 Topological Spaces	1
1.1 Basic Notions	1
1.2 Subspaces. Quotient Spaces	5
1.3 Products and Sums	8
1.4 Compact Spaces	11
1.5 Proper Maps	14
1.6 Paracompact Spaces	15
1.7 Topological Groups	15
1.8 Transformation Groups	17
1.9 Projective Spaces. Grassmann Manifolds	21
2 The Fundamental Group	24
2.1 The Notion of Homotopy	25
2.2 Further Homotopy Notions	30
2.3 Standard Spaces	34
2.4 Mapping Spaces and Homotopy	37
2.5 The Fundamental Groupoid	41
2.6 The Theorem of Seifert and van Kampen	45
2.7 The Fundamental Group of the Circle	47
2.8 Examples	52
2.9 Homotopy Groupoids	58
3 Covering Spaces	62
3.1 Locally Trivial Maps. Covering Spaces	62
3.2 Fibre Transport. Exact Sequence	66
3.3 Classification of Coverings	70
3.4 Connected Groupoids	72
3.5 Existence of Liftings	76
3.6 The Universal Covering	78
4 Elementary Homotopy Theory	81
4.1 The Mapping Cylinder	81
4.2 The Double Mapping Cylinder	84
4.3 Suspension. Homotopy Groups	86
4.4 Loop Space	89

4.5	Groups and Cogroups	90
4.6	The Cofibre Sequence	92
4.7	The Fibre Sequence	97
5	Cofibrations and Fibrations	101
5.1	The Homotopy Extension Property	101
5.2	Transport	107
5.3	Replacing a Map by a Cofibration	110
5.4	Characterization of Cofibrations	113
5.5	The Homotopy Lifting Property	115
5.6	Transport	119
5.7	Replacing a Map by a Fibration	120
6	Homotopy Groups	121
6.1	The Exact Sequence of Homotopy Groups	122
6.2	The Role of the Base Point	126
6.3	Serre Fibrations	129
6.4	The Excision Theorem	133
6.5	The Degree	135
6.6	The Brouwer Fixed Point Theorem	137
6.7	Higher Connectivity	141
6.8	Classical Groups	146
6.9	Proof of the Excision Theorem	148
6.10	Further Applications of Excision	152
7	Stable Homotopy. Duality	159
7.1	A Stable Category	159
7.2	Mapping Cones	164
7.3	Euclidean Complements	168
7.4	The Complement Duality Functor	169
7.5	Duality	175
7.6	Homology and Cohomology for Pointed Spaces	179
7.7	Spectral Homology and Cohomology	181
7.8	Alexander Duality	185
7.9	Compactly Generated Spaces	186
8	Cell Complexes	196
8.1	Simplicial Complexes	197
8.2	Whitehead Complexes	199
8.3	CW-Complexes	203
8.4	Weak Homotopy Equivalences	207
8.5	Cellular Approximation	210
8.6	CW-Approximation	211

8.7	Homotopy Classification	216
8.8	Eilenberg–Mac Lane Spaces	217
9	Singular Homology	223
9.1	Singular Homology Groups	224
9.2	The Fundamental Group	227
9.3	Homotopy	228
9.4	Barycentric Subdivision. Excision	231
9.5	Weak Equivalences and Homology	235
9.6	Homology with Coefficients	237
9.7	The Theorem of Eilenberg and Zilber	238
9.8	The Homology Product	241
10	Homology	244
10.1	The Axioms of Eilenberg and Steenrod	244
10.2	Elementary Consequences of the Axioms	246
10.3	Jordan Curves. Invariance of Domain	249
10.4	Reduced Homology Groups	252
10.5	The Degree	256
10.6	The Theorem of Borsuk and Ulam	261
10.7	Mayer–Vietoris Sequences	265
10.8	Colimits	270
10.9	Suspension	273
11	Homological Algebra	275
11.1	Diagrams	275
11.2	Exact Sequences	279
11.3	Chain Complexes	283
11.4	Cochain complexes	285
11.5	Natural Chain Maps and Homotopies	286
11.6	Chain Equivalences	287
11.7	Linear Algebra of Chain Complexes	289
11.8	The Functors Tor and Ext	292
11.9	Universal Coefficients	295
11.10	The Künneth Formula	298
12	Cellular Homology	300
12.1	Cellular Chain Complexes	300
12.2	Cellular Homology equals Homology	304
12.3	Simplicial Complexes	306
12.4	The Euler Characteristic	308
12.5	Euler Characteristic of Surfaces	311

13 Partitions of Unity in Homotopy Theory	318
13.1 Partitions of Unity	318
13.2 The Homotopy Colimit of a Covering	321
13.3 Homotopy Equivalences	324
13.4 Fibrations	325
14 Bundles	328
14.1 Principal Bundles	328
14.2 Vector Bundles	335
14.3 The Homotopy Theorem	342
14.4 Universal Bundles. Classifying Spaces	344
14.5 Algebra of Vector Bundles	351
14.6 Grothendieck Rings of Vector Bundles	355
15 Manifolds	358
15.1 Differentiable Manifolds	358
15.2 Tangent Spaces and Differentials	362
15.3 Smooth Transformation Groups	366
15.4 Manifolds with Boundary	369
15.5 Orientation	372
15.6 Tangent Bundle. Normal Bundle	374
15.7 Embeddings	379
15.8 Approximation	383
15.9 Transversality	384
15.10 Gluing along Boundaries	388
16 Homology of Manifolds	392
16.1 Local Homology Groups	392
16.2 Homological Orientations	394
16.3 Homology in the Dimension of the Manifold	396
16.4 Fundamental Class and Degree	399
16.5 Manifolds with Boundary	402
16.6 Winding and Linking Numbers	403
17 Cohomology	405
17.1 Axiomatic Cohomology	405
17.2 Multiplicative Cohomology Theories	409
17.3 External Products	413
17.4 Singular Cohomology	416
17.5 Eilenberg–Mac Lane Spaces and Cohomology	419
17.6 The Cup Product in Singular Cohomology	422
17.7 Fibration over Spheres	425
17.8 The Theorem of Leray and Hirsch	427

17.9	The Thom Isomorphism	431
18	Duality	438
18.1	The Cap Product	438
18.2	Duality Pairings	441
18.3	The Duality Theorem	444
18.4	Euclidean Neighbourhood Retracts	447
18.5	Proof of the Duality Theorem	451
18.6	Manifolds with Boundary	455
18.7	The Intersection Form. Signature	457
18.8	The Euler Number	461
18.9	Euler Class and Euler Characteristic	464
19	Characteristic Classes	467
19.1	Projective Spaces	468
19.2	Projective Bundles	471
19.3	Chern Classes	472
19.4	Stiefel–Whitney Classes	478
19.5	Pontrjagin Classes	479
19.6	Hopf Algebras	482
19.7	Hopf Algebras and Classifying Spaces	486
19.8	Characteristic Numbers	491
20	Homology and Homotopy	495
20.1	The Theorem of Hurewicz	495
20.2	Realization of Chain Complexes	501
20.3	Serre Classes	504
20.4	Qualitative Homology of Fibrations	505
20.5	Consequences of the Fibration Theorem	508
20.6	Hurewicz and Whitehead Theorems modulo Serre classes	510
20.7	Cohomology of Eilenberg–Mac Lane Spaces	513
20.8	Homotopy Groups of Spheres	514
20.9	Rational Homology Theories	518
21	Bordism	521
21.1	Bordism Homology	521
21.2	The Theorem of Pontrjagin and Thom	529
21.3	Bordism and Thom Spectra	535
21.4	Oriented Bordism	537
	Bibliography	541
	Symbols	551
	Index	557