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Buyalo, Sergei (RS-AOS2); **Schroeder, Viktor** (CH-ZRCH)
★**Elements of asymptotic geometry.**
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The present book is based on lectures by the authors and is roughly divided into two parts. The first part consists of eight chapters and gives an introduction to the theory of Gromov hyperbolic spaces. The second part with the remaining six chapters is devoted to various concepts of dimension of a metric space. A common theme in both parts of the book is embedding and non-embedding results. An appendix reviews various models for classical hyperbolic geometry.

Recall that a metric space X is called hyperbolic (in the sense of Gromov) if there exists $\delta \geq 0$ such that the inequality

$$|x-y|+|z-w| \leq \max\{|x-z|+|y-w|, |x-w|+|y-z|\}+\delta$$

holds for all points $x, y, z, w \in X$. It is a remarkable fact (discovered by Gromov) that many features of hyperbolic geometry can be developed in a general framework based on this simple inequality. A particular focus in this theory is the large scale or asymptotic geometry of the spaces. A natural class of mappings in this context is formed by quasi-isometries, i.e., maps $f\colon X \to Y$ between metric spaces X and Y for which there exist constants $\lambda \geq 1$, $C \geq 0$ such that

$$\frac{1}{\lambda}|u-v|-C \leq |f(u)-f(v)| \leq \lambda|u-v|+C$$

for all $u, v \in X$.

The first three chapters of the book present the basic results of this theory such as the stability of geodesics under quasi-isometries, the definitions of the boundary at infinity $\partial_\infty X$ of a hyperbolic space X and of its visual metrics on this boundary, and the concept of Busemann functions and their relation to "Gromov products". While this material is by now fairly standard, the authors' exposition is concise with an occasional elegant twist to the known proofs.

An important theme in hyperbolic geometry is the interplay between isometries of hyperbolic n-space $\mathbf{H}^n$ and Möbius transformations on the sphere at infinity $\mathbf{S}^{n-1} = \partial_\infty \mathbf{H}^n$. In a similar vein, there is a close correspondence of quasi-isometries between (geodesic) Gromov hyperbolic spaces and quasi-Möbius maps on their boundaries at infinity, i.e., maps that distort the metric cross-ratio

$$[a,b,c,d] = \frac{|a-c|\cdot|b-d|}{|a-b|\cdot|c-d|}$$

in a quantitatively controlled way. The authors give a painstaking analysis of this relation between maps on Gromov hyperbolic spaces and the induced maps on their boundaries in the next two chapters. Again this material is essentially known, but the presentation clarifies some subtleties that will be new even to experts. Thus, for example, the authors answer in a non-obvious way the question to which extent one has to assume that the Gromov hyperbolic spaces are geodesic in order to get induced maps with geometric control on their boundaries at infinity.

In the following two chapters, Chapters 6 and 7, a type of hyperbolic cone $\mathrm{Con}(Z)$

over an arbitrary metric space Z is introduced, and how classes of maps such as bi-Lipschitz, quasi-symmetric and quasi-Möbius maps between metric spaces extend to various classes of quasi-isometries between their hyperbolic cones is investigated.

Chapter 8 begins with a proof of P. Assouad's embedding theorem [Bull. Soc. Math. France **111** (1983), no. 4, 429–448; MR0763553]: if (Z,d) is a doubling metric space and $p\in(0,1)$, then one can embed the metric space (Z,d^p) into some $\mathbf{R}^n$ by a bi-Lipschitz map. The authors use this theorem together with results of the previous two chapters to prove an embedding theorem due to M. Bonk and O. Schramm [Geom. Funct. Anal. **10** (2000), no. 2, 266–306; MR1771428]: for every visual Gromov hyperbolic space of "bounded geometry" there exists a quasi-isometric embedding into some hyperbolic space $\mathbf{H}^n$.

In the next chapter the second part of the book starts and the focus switches to dimension theory. The authors introduce various notions of dimension of a metric space X. The definitions can be based on three basic quantities that one can associate with every open cover $\mathcal{U}$ of a metric space X: the multiplicity $m(\mathcal{U})$ is the maximal number of sets in $\mathcal{U}$ with non-empty intersection, the mesh of $\mathcal{U}$ is the supremum of the diameters of the sets in $\mathcal{U}$, and the Lebesgue number $L(\mathcal{U})$ is the supremum of all radii r such that every open ball in X of radius r is contained in some element of $\mathcal{U}$.

The topological dimension $\dim X$ of X can now be defined as the smallest number n such that for all for all $\epsilon>0$ there exists an open cover $\mathcal{U}$ of X with $m(\mathcal{U})\le n+1$ and $\mathrm{mesh}(\mathcal{U})<\epsilon$. One of the main concerns of the authors is to study the asymptotic dimension $\operatorname{asdim} X$ of a metric space X. It is the smallest number n with following property: for every (large) $R>0$ there exists an open cover $\mathcal{U}$ of X such that $m(\mathcal{U})\le n+1$, $\mathrm{mesh}(\mathcal{U})<\infty$, and $L(\mathcal{U})\ge R$. A related concept is the linearly controlled metric dimension $\operatorname{l-dim} X$, defined as the smallest number n such that there exists $\delta\in(0,1)$ so that for all sufficiently small $r>0$ there exists an open cover $\mathcal{U}$ of X with $m(\mathcal{U})\le n+1$, $\mathrm{mesh}(\mathcal{U})<r$, and $L(\mathcal{U})\ge\delta r$.

Other notions of dimensions studied by the authors are the Assouad-Nagata dimension and the linearly controlled asymptotic dimension. To give a unified treatment of all these concepts, the authors establish an axiomatic framework and show the expected basic results for these dimensions, such as monotonicity, product, and finite-union theorems. In this chapter, Chapter 9, the authors also formulate and prove Sperner's lemma, from which they deduce the fact that $\dim\mathbf{R}^n=n$.

In Chapter 10 the authors derive some relatively easy results on asymptotic dimension such as the lower bound

$$\operatorname{asdim} X\ge\dim\partial_\infty X+1$$

for any proper geodesic Gromov hyperbolic space X, or the equality

$$\operatorname{asdim}\mathbf{H}^2=2$$

which is obtained by embedding the hyperbolic plane in a product of two metric trees.

Next, Chapter 11 goes into a deeper analysis of the linearly controlled dimension and leads to some interesting consequences in Chapter 12: if X is a visual Gromov hyperbolic space metric space with $n:=\operatorname{l-dim}\partial_\infty X<\infty$, then there exists a quasi-isometric embedding of X into a product of $n+1$ metric trees. This result is due to the present authors [S. Buyalo, Algebra i Analiz **17** (2005), no. 4, 42–58; MR2173936; S. Buyalo and V. Schroeder, Geom. Dedicata **113** (2005), 75–93; MR2171299], as well as many other results discussed in these chapters. As a corollary they show that if X is a proper geodesic Gromov hyperbolic space X that is cobounded (there exists a bounded subset of X whose translates under the isometry group cover X), then

$$\operatorname{asdim} X=\dim\partial_\infty X+1.$$

Finally, the last two chapters discuss the concepts of hyperbolic dimension, hyperbolic rank and corank.

The exposition in this book is careful, with great attention paid to detail. It covers the basics of its subject, but also gives a presentation of more advanced subjects that appear in book form for the first time. *Mario Bonk*