Introduction

We shall study non-negative solutions u of the nonlinear evolution equations

$$\frac{\partial u}{\partial t} = \Delta \varphi(u) \quad x \in \mathbb{R}^n,\ 0 < t < T < +\infty \tag{1}$$

where the nonlinearity φ is assumed to be continuous, increasing, with $\varphi(0) = 0$ and satisfies the growth condition

$$a \leq \frac{u\,\varphi'(u)}{\varphi(u)} \leq a^{-1} \quad \text{for all } u > 0 \tag{2}$$

for some constant $a \in (0, 1)$ and the normalization condition $\varphi(1) = 1$. We shall denote the class of such nonlinearities φ by Γ_a.

The growth condition (2) is a natural generalization of the pure power case $\varphi(u) = u^m$, which is well known in the literature and arises in a number of physical problems. When $m = 1$ it reduces to the linear heat equation. When $m > 1$ equation (1) describes the flow of an isotropic gas through a porous medium (the porous medium equation), cf. [110]. Another application refers to heat radiation in plasmas [136]. When $m < 1$, equation (1) arises in the study of fast diffusions, in particular in models of gas-kinetics [35], [44], in diffusion in plasmas [21], and in thin liquid film dynamics driven by Van der Waals forces [64], [63]. Also it arises in geometry; the case $m = (n-2)/(n+2)$, in dimensions $n > 3$ describes the evolution of a conformal metric by the Yamabe flow [135] and it is related to the Yamabe problem, the case $m = 0$, $n = 2$ describes the Ricci flow on surfaces [77], [53], [133], and the case $m < 0$ in dimension $n = 1$ describes a plane curve shrinking along the normal vector with speed depending on the curvature [72], [68]. For a survey on the porous medium and fast diffusion equations see [111], [127].

The main objective of this book is to present the main results regarding the solvability of the Cauchy problem and the initial Dirichlet problem for equation (1) for a wide class of nonlinearities φ. The local regularity theory for equations of the form (1) will be presented as well. Special emphasis will be given to the various techniques, which although have been developed to study nonlinear equations of the form (1) may be applied to other nonlinear parabolic problems.

In the 1940s D. Widder [132] studied the characterization of the class of all non-negative solutions of the heat equation

$$\frac{\partial u}{\partial t} = \Delta u \quad \text{in } S_T = \mathbb{R}^n \times (0, T]. \tag{3}$$

In this case, the notion of solution is clear: $u \in L^1_{\text{loc}}(S_T)$ and the equation holds in the distribution sense (weak solution). It follows by classical regularity theorems that $u \in C^\infty(S_T)$.

The Widder theory can be expressed as follows. Let u be a non-negative weak solution of the heat equation in the strip S_T. Then:

(A.1) The solution u satisfies the growth condition

$$\sup_{0<t<T/2} \int u(x,t)\, e^{-C\,|x|^2}\, dx \; < \infty \tag{4}$$

where C is an absolute constant.

(A.2) There exists a non-negative Borel measure μ on $\mathbb{R}^n$ such that

$$\lim_{t\downarrow 0} u(\,\cdot, t) = d\mu \quad \text{in } D'(\mathbb{R}^n).$$

We shall call the measure μ the trace of u. Furthermore, the trace μ satisfies the growth condition

$$\int e^{-C\,|x|^2}\, d\mu \; < \infty \tag{5}$$

where C is an absolute constant.

(A.3) The solution u satisfies the pointwise estimate

$$u(x,t) \le C_t(u)\, e^{C\,|x|^2}$$

where C is an absolute constant and $C_t(u)$ depends on u and t.

(A.4) The trace μ determines the solution uniquely; if u, v are two non-negative weak solutions of equation (3) and

$$\lim_{t\downarrow 0} u(\,\cdot, t) = \lim_{t\downarrow 0} v(\,\cdot, t)$$

then $u \equiv v$.

(A.5) For each non-negative Borel measure μ on $\mathbb{R}^n$ satisfying the growth condition (5) there is a non-negative continuous weak solution u of (3) in $S_{T/M}$ with trace μ, and

$$u(x,t) = \frac{C_n}{t^{n/2}} \int e^{-|x-y|^2/4t}\, d\mu(y)$$

for an absolute constant C_n depending only on dimension n.

Let us note that the assumption that the class consists of non-negative solutions is necessary for the Widder theory to hold true, as there are specific examples of oscillating solutions with initial data identically equal to zero which do not satisfy condition (4).

The porous medium equation, i.e. $\varphi(u) = u^m$, $m > 1$, in (1) has been studied extensively and is by now well understood. In fact, by combining the results of Aronson and Caffarelli [8], Bénilan, Crandall, and Pierre [20] and Dahlberg and Kenig [45] one obtains the complete analogue of the Widder theory for this case. Let u be a non-negative continuous distributional solution of the equation

$$\frac{\partial u}{\partial t} = \Delta u^m, \quad m > 1, \ (x,t) \in S_T. \tag{6}$$

Then:

(B.1) The spatial averages of the solution u satisfy the growth condition

$$\sup_{0<t<T} \sup_{R>1} \frac{1}{R^{n+2/(m-1)}} \int_{|x|<R} u(x,t)\,dx < \infty.$$

(B.2) The initial trace μ exists; for any continuous distributional solution u of (1) there exists a Borel measure μ such that

$$\lim_{t\downarrow 0} u(\cdot,t) = d\mu \quad \text{in } D'(\mathbb{R}^n)$$

and satisfies the growth estimate

$$\sup_{R>1} \frac{1}{R^{n+2/(m-1)}} \int_{|x|<R} d\mu < \infty. \tag{7}$$

(B.3) The solution u satisfies the pointwise growth estimate

$$u^{m-1}(x,t) \le C_t(u)\,(1+|x|^2) \quad \text{for } t \in (0, T/2)$$

where $C_t(u) = C_T\,(u(0,T))\,t^{-\lambda}$, $\lambda = n/(2+n(m-1))$ as $t \downarrow 0$.

(B.4) The trace μ determines the solution uniquely; if u, v are two non-negative continuous distributional solutions and

$$\lim_{t\downarrow 0} u(\cdot,t) = \lim_{t\downarrow 0} v(\cdot,t)$$

then $u \equiv v$.

(B.5) For every measure μ on $\mathbb{R}^n$ satisfying (7) there exists a non-negative continuous distributional solution of (6) with trace μ satisfying (7).

Let us note that the assumption on the continuity of u is not essential, due to the result of Dahlberg and Kenig in [49] where it is shown that if $u \in L^m_{\mathrm{loc}}(\Omega)$, $u \ge 0$ and $\partial u/\partial t = \Delta u^m$ in $D'(\Omega)$, then u is continuous.

In order to extend the above results from the porous medium equation $\varphi(u) = u^m$, $m > 1$ to the case of equation (1) we consider the class $\mathcal{S}_a$ of nonlinearities φ, corresponding to *slow diffusion*, which is defined by the following conditions:

(i) $\varphi\colon [0,\infty) \to \mathbb{R}$ is continuous, strictly increasing with $\varphi(0) = 0$;

(ii) there exist $a \in (0, 1)$ such that for any $u > 0$

$$a \leq \frac{u\,\varphi'(u)}{\varphi(u)} \leq \frac{1}{a} \quad \text{(polynomial growth);}$$

(iii) there exists $u_0 > 0$ such that and for any $u \geq u_0$

$$1 + a \leq \frac{u\,\varphi'(u)}{\varphi(u)} \quad \text{(super-linearity);}$$

(iv) $u_0 = 1$ and $\varphi(1) = 1$ (normalization).

The condition (iv) is only technical and imposed to normalize the class $\mathcal{S}_a$. Conditions (ii) and (iv) imply the pointwise polynomial growth condition

$$u^{1/a} \leq \varphi(u) \leq u^a \quad \text{for } 0 \leq u \leq 1$$

while conditions (iii) and (iv) imply the pointwise super-linear polynomial growth condition

$$u^{1+a} \leq \varphi(u) \leq u^{1/a} \quad \text{for } u \geq 1.$$

It is clear that the porous medium equation $\varphi(u) = u^m$, $m > 1$, belongs to $\mathcal{S}_a$. However, the super-linear growth on $\varphi(u) \in \mathcal{S}_a$ is only assumed for large values of u.

The Widder theory in this case can be described as follows: by hypothesis $\psi(u) = \varphi(u)/u$, for $u \geq 1$ is an increasing function, so that we may define $\Lambda(u) \equiv \psi^{-1}(u)$. Thus if u is a non-negative continuous weak solution of (1) then the following holds:

(C.1) Growth condition (B.1) holds with $\Lambda(R^2)$ instead of $R^{2/(m-1)}$.

(C.2) Condition (B.2) holds with $\Lambda(R^2)$ instead of $R^{2/(m-1)}$ in the growth estimate (7) on the initial trace.

(C.3) The pointwise estimate (B.3) holds with $\psi(u)$ instead of u^{m-1}.

(C.4) and (C.5) are similar to (B.4) and (B.5).

Following the ideas in [49] one can remove the assumption on the continuity of u in the case where φ is convex. For general φ in the class $\mathcal{S}_a$ an analogous result remains an open problem.

Consider next equation (1) with nonlinearity φ which is either a power $\varphi(u) = u^m$ with $(n-2)/n < m < 1$ or, more generally, it belongs to the class $\mathcal{F}_a$, corresponding to super-critical fast diffusion, which is defined by the following conditions:

(i) $\varphi\colon [0, \infty) \to \mathbb{R}$ is continuous, strictly increasing with $\varphi(0) = 0$;

(ii) there exists $a \in (0, 1)$ such that for any $u > 0$

$$a \leq \frac{u\,\varphi'(u)}{\varphi(u)} \leq \frac{1}{a} \quad \text{(polynomial growth);}$$

(iii) there exists $u_0 > 0$ such that for $u \geq u_0$

$$\frac{n-2}{n} + a \leq \frac{u\,\varphi'(u)}{\varphi(u)} \leq 1 - a \quad \text{(sublinearity);}$$

(iv) $u_0 = 1$ and $\varphi(1) = 1$ (normalization).

Conditions (ii) and (iv) imply the pointwise polynomial growth condition

$$u^{1/a} \leq \varphi(u) \leq u^a \quad \text{for } 0 \leq u \leq 1$$

while conditions (iii) and (iv) imply the pointwise sublinear (since $1 - a < 1$) and super-critical (since $(n-2)/n + a > (n-2)/n$) polynomial growth condition

$$u^{\frac{n-2}{n}+a} \leq \varphi(u) \leq u^{1-a} \quad \text{for } u \geq 1.$$

The above growth conditions generalize the fast diffusion equation $\varphi(u) = u^m$, in the super-critical range of exponents $(n-2)/n < m < 1$, which in particular belongs to the class $\mathcal{F}_a$.

It follows by the results of Herrero and Pierre [81], and Dahlberg and Kenig [47] that in the fast diffusion case no growth conditions need to be imposed on the initial trace for existence, as described next:

(D.1) For any non-negative continuous distributional solution u of (1), there exists a unique locally finite Borel measure μ on $\mathbb{R}^n$ such that

$$\lim_{t \downarrow 0} \int_{\mathbb{R}^n} u(x,t)\,\psi(x)\,dx = \int_{\mathbb{R}^n} \psi(x)\,d\mu(x)$$

for all $\psi \in C_0^\infty(\mathbb{R}^n)$.

(D.2) The trace μ determines the solution uniquely: if u, v are two non-negative continuous distributional solutions and

$$\lim_{t \downarrow 0} u(\,\cdot\,,t) = \lim_{t \downarrow 0} v(\,\cdot\,,t)$$

then $u \equiv v$.

(D.3) For any locally finite Borel measure μ on $\mathbb{R}^n$ there exists a continuous distributional solution u of (1) in $S_\infty = \mathbb{R}^n \times (0,\infty)$ with trace μ.

(D.4) For any non-negative continuous distributional solution u of (1) in S_T, there exists a non-negative continuous distributional solution $\hat{u}$ of (1) in S_∞ with $u = \hat{u}$ in S_T.

The super-critical assumption (iii) is essential for the theory described above. Indeed, in the sub-critical case $m \leq (n-2)/n$ the analogues of the above results do not hold true. In particular, there is no continuous distributional solution of equation

$u_t = \Delta u^m$, $m \le (n-2)/n$ with initial data the Dirac mass. We refer the reader to Section 3.3 for details.

One particularly interesting case of fast diffusion is the case where $\varphi(u) = \log u$, corresponding to the limiting case of $\varphi(u) = u^m$, when $m \to 0$. It has been shown by Esteban, Rodríguez and Vazquez in [69], and Daskalopoulos and del Pino in [52] that a strong non-uniqueness phenomenon takes place in this case. In the critical dimension $n = 2$ this phenomenon is related to the topological properties of solutions to the Ricci flow, corresponding to evolving metrics on compact surfaces, non-compact surfaces and orbifolds. We refer the reader to Section 3.2 of Chapter 3 for the details on the solvability and well-posedness of the Cauchy problem for the logarithmic fast diffusion equation.

A brief outline of the contents of the book is as follows.

In Chapter 1 we shall collect a series of preliminary, yet very important results, concerning continuous distributional solutions of equation (1). These results will be used throughout the book. We emphasize the a priori L^∞ bounds, the Harnack inequality for solutions of slow diffusion, and the equicontinuity of solutions.

Chapter 2 deals with the solvability of the Cauchy problem for equation (1) in the slow diffusion case $\varphi \in \mathcal{S}_a$. We shall provide a complete characterization of non-negative weak solutions of (2.0.1) in terms of their initial condition, showing in particular the results (B.1)–(B.5) and their extensions (C.1)–(C.5).

The first part of Chapter 3 is devoted to the solvability of the Cauchy problem for equation (1) in the super-critical fast diffusion case $\varphi \in \mathcal{F}_a$. We shall present a theory which completely classifies the class of continuous weak solutions of (3.1.1) in terms of their initial condition, showing in particular (D.1)–(D.4). The second part of this chapter is devoted to the Cauchy problem for the logarithmic fast diffusion equation $u_t = \Delta \log u$, in dimensions $n \ge 2$. We give special emphasis to the critical case $n = 2$, where many interesting phenomena can be observed with important geometric applications. We show that in this case a strong non-uniqueness phenomenon takes place and establish the existence a continuum of solutions with a given initial data. In the last section of this chapter we comment on the solvability and well-posedness of equation $u_t = \Delta u^m$ in the sub-critical case $0 < m \le (n-2)/n$ as well as in the super-fast diffusion case $m < 0$.

In Chapter 4 we study the class of non-negative strong solutions of the initial Dirichlet problem for equation (1) on $D \times (0, \infty)$, $D \subset \mathbb{R}^n$ open bounded, in the slow diffusion case $\varphi \in \mathcal{S}_a$. We establish the existence of an exceptional solution α with infinite initial data. We then show that any other strong solution of the initial Dirichlet problem is uniquely characterized by its initial traces μ and λ, namely non-negative Borel measures that are supported on D and ∂D respectively. We also study in this chapter the initial Dirichlet problem for equation (1) in the pure power fast diffusion case, $\varphi(u) = u^m$, in the range $(n-2)_+/n < m < 1$.

Our last Chapter 5 is devoted to the study of the regularity properties of weak solutions to the porous medium equation $u_t = \Delta u^m$, $m > 1$. We establish that weak solutions are continuous.

The last section in each chapter is devoted to a brief summary of further known results as well as several open problems related to the theory presented.

Denote by Γ_a the class of nonlinearities φ which satisfy the following conditions:

(i) $\varphi\colon [0, \infty) \to \mathbb{R}$ is a continuous non-negative function with $\varphi(0) = 0$;

(ii) there exist a constant $a \in (0, 1)$ such that

$$a \leq \frac{u\,\varphi'(u)}{\varphi(u)} \leq a^{-1} \quad \text{for all } u > 0;$$

(iii) $\varphi(1) = 1$ (normalization).

For future reference, we close the Introduction with the statement of three results, proved in following chapters, which will play a fundamental role throughout the book.

The following compactness result due to P. Sacks [118] will be proved in Section 1.5 of Chapter 1, and will be used extensively throughout the book.

Theorem H.1. *Let $\{u_k\}$ be a sequence of continuous non-negative distributional solutions of the equation $\partial u/\partial t = \Delta\varphi(u)$ in R, with $\varphi \in \Gamma_a$. If $\{u_k\}$ is uniformly bounded in R, then the u_k's are equicontinuous in S.*

In the case that $\varphi \in \mathcal{S}_a$, the following stronger result due to DiBenedetto and Friedman [67] holds: *Let u be a non-negative continuous distributional solution of the equation $u_t = \Delta\varphi(u)$ in a compact domain $R \subset \mathbb{R}^n \times (0, \infty)$, with $\varphi \in \mathcal{S}_a$. Then*

$$\left|u(x, t) - u(x', t')\right| \leq C(M)\ \left\{|x - x'| + |t - t'|\right\}^{\alpha}$$

for any $(x, t), (x', t') \in S \subset\subset R$, where $\alpha = \alpha(a, n)$ and the constant M is given by $M = \sup_t \int_R u(x, t)\,dx$.

In other words, continuous solutions in R are Hölder continuous in S. This result is sharp, i.e. examples show that u need not be more regular than Hölder continuous. To see this consider the Barenblatt self-similar solution of the Cauchy problem

$$\begin{aligned} u_t &= \Delta u^m \quad \text{in} \mathbb{R}^n \times \mathbb{R}^+, \\ u(x, 0) &= \delta(0) \quad x \in \mathbb{R}^n, \end{aligned}$$

which has the explicit form

$$B(x, t) = t^{-\alpha} \left[\left(M - k\,\frac{|x|^2}{t^{\beta}}\right)_+\right]^{\frac{1}{m-1}}$$

with

$$\alpha = \frac{n}{n\,(m-1)+2}, \quad \beta = \frac{1}{n(m-1)+2}$$

and M, k specific constants which depend only on m, n. It is clear that for $m > 2$, the solution $B(x, t)$ is Hölder continuous with exponent $\alpha \leq \alpha_0(m, n)$ but not Lipschitz.

We shall use the following weaker version of (C.5) which will be proved in Section 1.6 of Chapter 1:

Theorem H.2. *Let $f \in L^1(\mathbb{R}^n)$ be a non-negative function. Then there exists a unique non-negative continuous distributional solution of the equation* (1) *in $\mathbb{R}^{n+1}_+$ such that*

$$\sup_{t>0} \int u(x,t)\,dx \le \int f(x)\,dx$$

and

$$\lim_{t \downarrow 0} \|u(\cdot,t) - f(x)\|_{L^1(\mathbb{R}^n)} = 0.$$

Furthermore if f is radially decreasing so is $u(\cdot,t)$, for each $t > 0$.

The following uniqueness result due to M. Pierre [113] will be proved in Section 2.4 of Chapter 2:

Theorem H.3. *If u_1 and u_2 are continuous non-negative distributional solutions of the equation* (1) *with $\varphi \in \Gamma_a$ such that*

$$\sup_{t>0} \int (u_1(x,t) + u_2(x,t))\,dx < \infty$$

with

$$u_1, u_2 \in L^\infty(\mathbb{R}^n \times [\tau, \infty)) \quad \textit{for each } \tau > 0$$

and

$$\lim_{t \downarrow 0} u_1(\cdot,t) = \lim_{t \downarrow 0} u_2(\cdot,t) \quad \textit{in } D'(\mathbb{R}^n)$$

then $u_1 = u_2$.