Preface

A single book is certainly not enough to describe the rich and historical relationships between physics and number theory. This volume presents a selection of problems which are currently in full development and inspire the research of many people. All the papers begin with a survey which will make it possible even for non-specialists to understand them and will give an idea of the great variety of subjects and techniques in this frontier area.

The first paper, "The phase of oscillations and prime numbers: classical and quantum", by Michel Planat, is an example of the strong connection between physics and mathematics. It starts from a concrete problem and brings into play an impressive variety of mathematical techniques, especially in number theory. The paper provides an accessible introduction to the problem of phase-locking in oscillating systems, both at a classical level and at a quantum level. The mathematical formulation of the different aspects of this problem requires numerous tools: first, you see how prime numbers appear, together with continuous fractions and the Mangoldt function. Then, come some hyperbolic geometry and the Riemann ζ-function. On the quantum side, roots of unity and Ramanujan sums are related to noise in oscillations, and when discussing phase in quantum information, the author uses Bost and Connes KMS states, Galois rings and fields along with some finite projective geometry.

Next there are two papers about crystallography. From a physical point of view, a crystal is a solid having an essentially discrete diffraction diagram. It can be periodic or not. From a mathematical point of view, lattices in $\mathbb{R}^n$ are good tools to describe periodic crystals, but not aperiodic ones. Very little is known about aperiodic crystals, apart from the so called *quasicrystals*, whose diffraction diagrams present some regularity: they are invariant under dilatation by a factor that may be irrational. They are well described by some discrete sets called *cut-and-project* sets, which are a generalisation of lattices. In his paper "On Self-Similar Finitely Generated Uniformly Discrete (SFU-)Sets and Sphere Packings", Jean-Louis Verger-Gaugry is interested in cut-and-project sets in $\mathbb{R}^n$. The first part of the paper is a survey of the link between the geometry of numbers and aperiodic crystals in physics, from the mathematical point of view. In the second part, the author proves some new results about the distances between the points of cut-and-project sets. By considering each point as the centre of a sphere, one gets a sphere packing problem which is hopefully a good model for atom packing. In "Nested quasicrystalline discretisation of the line", Jean-Pierre Gazeau, Zuzana Masáková, and Edita Pelantová focus on cut-and-project sets obtained from a square lattice in $\mathbb{R}^2$, with the idea of constructing aperiodic wavelets. They review the geometrical properties of such sets, their combinatorial properties from the point of view of language theory and their relation to nonstandard numeration systems based on θ when the cut-and-project set is self-similar for an irrational scaling factor θ. Finally, they provide an algorithm which generates cut-and-project sets.

In "Hopf algebras in renormalization theory: locality and Dyson–Schwinger equations from Hochschild cohomology", Christoph Bergbauer and Dirk Kreimer reformulate a problem from quantum field theory in algebraic terms. In the first part, the authors explain how, by constructing an appropriate Hopf algebra structure on rooted trees, one gets an algebraic formulation of the renormalization process for Feynman graphs. After an introductory overview of those Hopf algebras, they give numerous examples of their generalizations. In the second part, they recall the definition of the Hochschild cohomology associated to such Hopf algebras. A physical interpretation of the cocycles leads to the Dyson–Schwinger equations, which describe the loop expansion of Green function in a recursive way. This method is a straightforward alternative to the usual one. The paper ends up with a description of the Dyson–Schwinger equation from different points of view: it provides transcendental numbers, it is a tool to define a generating function for the polylogarithm and it is the equation of motion for a renormalizable quantum field theory.

"Fonction ζ et matrices aléatoires" by Emmanuel Royer, is an extensive survey on the zeroes of L-functions. The analytic properties of the Riemann ζ-function describe the behaviour of prime numbers. More generally, one can attach an L-function to certain arithmetic or geometric objects, and thus obtain some information on them by analytic methods. The functions described in this paper are the Riemann ζ-function and L-functions attached to Dirichlet characters, to automorphic representations, or to modular forms. The Generalised Riemann Hypothesis predicts that all the zeroes of those L-functions lie on the line $\mathrm{Re}(z) = \frac{1}{2}$, and other important conjectures relate their statistical behaviour along this line to the statistical behaviour of the spectrum of large unitary matrices.

The paper by Philippe Michel, "Some recent applications of Kloostermania", is devoted to Kloosterman sums which are a special kind of algebraic exponential sums. They where first developed to bound the number of representations of a large integer n by a diagonal quaternary definite quadratic form

$$n = ax_1^2 + bx_2^2 + cx_3^2 + dx_4^2.$$

They eventually turned out to be one of those fascinating objects with two faces: the Petersson–Kuznetsov trace formula relates sums of Kloostermans sums, data of an arithmetico-geometric nature, to the Fourier coefficients of modular forms, data of a spectral nature. First written by Kuznetsov for $\mathrm{SL}_2(\mathbb{Z})$ the formula was extended by Deshouillers and Iwaniec to arbitrary congruence subgroups. The use of this connection in both directions is a powerful tool in analytic number theory. There are many applications. For example, one can get good estimates of linear combinations of Fourier coefficents, or get an upper bound for the dimension of the space of weight one modular forms, or solve many instances of the subconvexity problem. The last application presented in the paper is that, using the Petersson–Kuznetzov trace formula, one can refine the error term in Weyl's law and that these refinements can be interpreted in terms of Quantum Chaos.

Finally, Ariane Mézard's "Introduction à la correspondance de Langlands locale", is a survey of the local Langlands correspondence. When K is a non archimedian local field, and W_K its Weil group, the local class field isomorphism

$$W_K^{ab} \simeq K^*$$

can be interpreted as a bijection between continuous irreducible representations of $K^* = \mathrm{GL}_1(K)$ and one dimensional continuous representation of W_K. The Langlands conjecture is a generalisation of this isomorphism in higher dimension: for any $n \geq 1$ there should be an isomorphism between isomorphism classes of continuous irreducible admissible representations of $\mathrm{GL}_n(K)$ and isomorphism classes of n-dimensional Φ-semi-simple continuous representations of the Weil–Deligne group W'_K. In her paper, Ariane Mézard studies the bijection between isomorphism classes of irreducible admissible representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ over $\overline{\mathbb{Q}}_\ell$, and isomorphism classes of some representations of $W_{\mathbb{Q}_p}$ in $\mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$. She explains the main steps of the proof in the ℓ-adic case (this means $\ell \neq p$, in which case the conjecture is proved for any n). She also points out where and why problems arise in the p-adic case ($\ell = p$). She describes the new objects and explains the strategy for $n = 2$. The p-adic conjecture is still open.

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Strasbourg, August 2006 *Louise Nyssen*