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★Differential Harnack inequalities and the Ricci flow.

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This book is a nice introduction to the Ricci flow, with a focus on differential Harnack inequalities, entropy formulas and G. Perelman's reduced volume functional. The author makes many comparisons with the standard heat equation on a fixed manifold and relates results to Ricci solitons where appropriate. The book is a very readable starting point for exploration of this exciting area of current research in nonlinear PDE and differential geometry.

Outline of the book:

"The book is divided into four main chapters. In chapter one we explain general variations of Riemannian metrics and introduce the Ricci flow as the (weakly) parabolic part of the L^2 -gradient flow of the Einstein-Hilbert functional. We then prove that there exists no functional which has the Ricci flow as its gradient flow (Proposition 1.7). After computing various evolution equations for the Ricci flow, we introduce gradient Ricci solitons and derive equations for the three cases of steady, shrinking and expanding solitons (Propositions 1.14, 1.15 and 1.16). A reader who is familiar with Hamilton's papers will already know most of the results in this chapter.

"Chapter two is devoted to the study of differential Harnack inequalities. We start with the famous Li-Yau Harnack estimate, which we present in its original form (Proposition 2.5) as well as in a quadratic and an integrated version (Corollaries 2.6 and 2.7). We then explain the maximum principle for systems to prove Hamilton's matrix Harnack inequality for the heat equation (Proposition 2.11). Finally, we proceed with Hamilton's Harnack inequalities for the Ricci flow (Theorem 2.14), where we only give a heuristic motivation instead of rewriting the rather lengthy proof.

"In the third chapter we present Lei Ni's entropy formulas for the heat equation on a static manifold. The main result (Theorem 3.8) will relate these entropy formulas with the Nash entropy and the Li-Yau Harnack inequality. Moreover, a local version of an entropy formula for the positive heat kernel (Proposition 3.6) will lead us to a new Li-Yau type differential Harnack inequality, which gives a lower bound for the heat kernel when being integrated (Corollary 3.10). We also discuss the entropy formulas for steady, shrinking and expanding solitons, the most important being the shrinking case in which we will again integrate a Harnack type inequality (Proposition 3.15) to find a lower bound for the adjoint heat kernel under Ricci flow (Corollary 3.16), as in Section 9 of Perelman's paper ["The entropy formula for the Ricci flow and its geometric applications", preprint, arxiv.org/abs/math/0211159]. The quantities $l(q, \bar{\tau})$ and $l(q, T)$ which one finds in these two corollaries 3.16 and 3.10, respectively, will turn out to be Perelman's backwards reduced distance and its analog for the heat kernel on a static manifold.

"This motivates our investigation of the two corresponding distance-functionals in chapter four. We first examine the static case, where the computations are much easier. This will lead to Proposition 4.2, a new result for the heat kernel on a manifold, which coincides with Perelman's result (Theorem 4.9) in the case of a Ricci flat manifold. The second part of chapter four provides a detailed exposition of Perelman's \mathcal{L} -length, \mathcal{L} -

geodesics and the \mathcal{L} -exponential map. We finish with the monotonicity of Perel'man's backwards reduced volume (Corollary 4.13).”

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